AN ANALOG OF THE LUSIN-PRIVALOFF RADIAL UNIQUENESS THEOREM

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1. Let $D=\{|z|<1\}$ and $C=\{|z|=1\}$. For $0\leqq\theta<2\pi$, and $-\pi/2<\phi<\pi/2$, let $\rho(\theta)=\{re^{i\theta}:0\leqq r<1\}$ and $\rho(\theta,\ \phi)=\{e^{i\theta}(1-re^{i\phi}):0<r\leqq\cos\phi\}$. If $f(z)$ is a meromorphic function in D, and $G\subseteq D$ such that $e^{i\theta}\in\overline{G}\cap C$, then $C_G(f,\ e^{i\theta})$ represents the cluster set of $f(z)$ at $e^{i\theta}$ restricted in G. $F(f)$ will denote the set of Fatou points of $f(z)$ on C. (For a definition of Fatou points, see [**6**, p. 61].) Finally, a subset S of C is said to be metrically dense on an arc A of C if meas $(A'\cap S)>0$ for each subarc A' of A.

Barth and Schneider have proved the following analog of the F. and M. Riesz uniqueness theorem for bounded holomorphic functions.

THEOREM A [**3**, THEOREM 1]. *Let μ be a decreasing function on* $[0,\ 1)$ *such that* $\lim_{r\to 1}\mu(r)=0$, *and let S be a subset of C of second category. If $f(z)$ is a bounded holomorphic function in D and* $|f(re^{i\theta})|=o(\mu(r))$ *for each $e^{i\theta}$ in S, then* $f(z)\equiv 0$.

In Remark 4 of [**3**], the question whether the above theorem holds for holomorphic functions has been raised. We shall show that it even holds for meromorphic functions.

THEOREM. *Let μ and S be the same as stated in Theorem* A. *If $f(z)$ is a meromorphic function in D and* $|f(re^{i\theta})|=o(\mu(r))$ *for each $e^{i\theta}$ in S, then* $f(z)\equiv 0$.

Before showing the proof, we wish to remark that our theorem is an *analog* of the Lusin-Privaloff uniqueness theorem which states:

If $f(z)$ is a holomorphic function in D and $\lim_{r\to 1}f(re^{i\theta})=0$ *for each $e^{i\theta}$ in S, S being both metrically dense and of second category on an arc A of C, then* $f(z)\equiv 0$.

2. To prove our theorem, we need two lemmas. The first one is very elementary in applying the notion of Baire categories. The second one is an extension of [**4**, Lemma 1].

LEMMA. 1. *Let S be a set of second category in an interval J, then there exists a subinterval I of J such that $S\cap I$ is both dense and of second category in I.*

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PROOF. The existence of an interval in which S is of second category and of an interval in which S is dense are immediate consequences of the definition of "second category." We must prove the existence of an interval with both of these properties. By the remark following the proof of [**5**, Theorem 35, p. 201], there exists a nonempty closed subinterval I' of J such that the intersection of S with any neighborhood N of a point in I' is of second category in I'. Thus a subinterval I of I' in which S is dense has all the properties required.

Applying Lemma 1, it is easy to see that we can modify the proof of [**4**, Lemma 1, pp. 170–171] and obtain

LEMMA 2. *Suppose that $f(z)$ is a meromorphic function in D and that for some fixed ϕ, $-\pi/2<\phi<\pi/2$, and some complex number β, finite or infinite, there exists a set S of second category on C and such that*

$$\beta \notin \bigcup_{e^{i\theta}\in S} C_{\rho(\theta,\phi)}(f, e^{i\theta}).$$

Then there exists an arc A on C such that

(1) *Either $f(z)$ or $1/(f(z)-\beta)$ is uniformly bounded in a relative neighborhood of A in D according as $\beta=\infty$ or $\beta\neq\infty$,*

(2) *$A\cap S$ is dense in A, and*

(3) *$A\cap S$ is of second category in A.*

3. We now proceed to prove our theorem. Since $\lim_{r\to 1}\mu(r)=0$ and $|f(re^{i\theta})|=o(\mu(r))$ for each $e^{i\theta}$ in S, we have

$$\infty \notin \bigcup_{e^{i\theta}\in S} C_{\rho(\theta)}(f, e^{i\theta}).$$

Note that S is of second category. By Lemma 2, there exists an arc A of C such that $f(z)$ is uniformly bounded in a relative neighborhood of A in D and that $A\cap S$ is of second category in A.

On the other hand, by a theorem [**2**, p. 6] of Barth and Schneider, there exists a *nonconstant* holomorphic function $g(z)$ in D such that

$$\max_{0\leq\theta<2\pi} |g(re^{i\theta})| < 1/\mu(r) \quad \text{and} \quad \lim_{r\to 1} g(re^{i\theta}) = 0$$

for each $e^{i\theta}$ in T, where T is a subset of C of measure 2π. Now, consider the function $h(z)=f(z)g(z)$. Note that

$$\lim_{r\to 1} h(re^{i\theta}) = 0$$

for each $e^{i\theta}$ in $(S\cup T)\cap A$. Since $(S\cup T)\cap A$ is both metrically dense and of second category in A, $h(z)$ satisfies the hypotheses of [**4**, Theorem 1] and we have $h(z)\equiv 0$. Hence $f(z)\equiv 0$.

4. Gauthier (in a written communication to the author) has kindly pointed out that our theorem is sharp in the following sense:

THEOREM B. *Let μ be as in Theorem* A *and S be any set of first category on C. Then there exists a nonconstant holomorphic function $f(z)$ in D such that $|f(re^{i\theta})| = o(\mu(r))$ for each $e^{i\theta}$ in S.*

PROOF. Since S is of first category, $S = \bigcup_{n=1}^{\infty} S_n$, where S_n, for each $n = 1, 2, \cdots$, is nowhere dense on C. Thus $S' = \bigcup_{n=1}^{\infty} \overline{S}_n$ is also of first category on C and $S \subseteq S'$. Consider the set

$$E' = \bigcup_{n=1}^{\infty} \{re^{i\theta} : e^{i\theta} \in \overline{S}_n,\ 1 - 1/2n \leqq r < 1\}.$$

Choose θ_0 such that $e^{i\theta_0} \notin S'$ and let $E = E' \cup \rho(\theta_0)$. Note that E is relatively closed and nowhere dense in D. We define

$$\begin{aligned} \phi(z) &= 0, \qquad z \in E', \\ &= 1, \qquad z \in \rho(\theta_0). \end{aligned}$$

$\phi(z)$ is continuous on E. Hence by a theorem of Arakeljan (see Arakeljan [1]), there exists a holomorphic function $f(z)$ in D such that $|f(z) - \phi(z)| \leqq \mu(|z|)(1 - |z|)$ for each z in E. It follows that $|f(re^{i\theta})| = o(\mu(r))$ for each $e^{i\theta}$ in S' (note that $S \subseteq S'$), and $f(z) \not\equiv 0$ because $f(z)$ is near to one on $\rho(\theta_0)$.

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REFERENCES

1. N. U. Arakeljan, *Uniform and tangential approximation with analytic functions*, Izv. Akad. Nauk Armjan. SSR Ser. Mat. **3** (1968), no. 45, 273–286. (Russian)

2. K. F. Barth and W. J. Schneider, *On the impossibility for extending the Riesz uniqueness theorem to functions of slow growth*, Ann. Acad. Sci. Fenn. Ser. AI **432** (1968), 9 pp.

3. ———, *An asymptotic analog of the F. and M. Riesz radial uniqueness theorem*, Proc. Amer. Math. Soc. **22** (1969), 53–54.

4. E. F. Collingwood, *On the linear and angular cluster sets of functions meromorphic in the unit circle*, Acta Math. **91** (1954), 165–185. MR **16**, 460.

5. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955. MR **16**, 1136.

6. K. Noshiro, *Cluster sets*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Heft 28, Springer-Verlag, Berlin and New York, 1960. MR **24** #A3295.

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