AN ANALOG OF THE LUSIN-PRIVALOFF RADIAL UNIQUENESS THEOREM

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1. Let $D = \{ |z| < 1 \}$ and $C = \{ |z| = 1 \}$. For $0 \leq \theta < 2\pi$, and $-\pi/2 < \phi < \pi/2$, let $\rho(\theta) = \{ re^{i\theta} : 0 \leq r < 1 \}$ and $\rho(\theta, \phi) = \{ e^{i\theta}(1 - re^{i\phi}) : 0 < r \leq \cos \phi \}$. If $f(z)$ is a meromorphic function in $D$, and $G \subseteq D$ such that $e^{i\theta} \in \overline{G} \cap C$, then $C_\theta(f, e^{i\theta})$ represents the cluster set of $f(z)$ at $e^{i\theta}$ restricted in $G$. $F(f)$ will denote the set of Fatou points of $f(z)$ on $C$. (For a definition of Fatou points, see [6, p. 61].) Finally, a subset $S$ of $C$ is said to be metrically dense on an arc $A$ of $C$ if $\text{meas}(A' \cap S) > 0$ for each subarc $A'$ of $A$.

Barth and Schneider have proved the following analog of the F. and M. Riesz uniqueness theorem for bounded holomorphic functions.

Theorem A [3, Theorem 1]. Let $\mu$ be a decreasing function on $[0, 1)$ such that $\lim_{r \to 1} \mu(r) = 0$, and let $S$ be a subset of $C$ of second category. If $f(z)$ is a bounded holomorphic function in $D$ and $|f(re^{i\theta})| = o(\mu(r))$ for each $e^{i\theta}$ in $S$, then $f(z) \equiv 0$.

In Remark 4 of [3], the question whether the above theorem holds for holomorphic functions has been raised. We shall show that it even holds for meromorphic functions.

Theorem. Let $\mu$ and $S$ be the same as stated in Theorem A. If $f(z)$ is a meromorphic function in $D$ and $|f(re^{i\theta})| = o(\mu(r))$ for each $e^{i\theta}$ in $S$, then $f(z) \equiv 0$.

Before showing the proof, we wish to remark that our theorem is an analog of the Lusin-Privaloff uniqueness theorem which states:

If $f(z)$ is a holomorphic function in $D$ and $\lim_{r \to 1} f(re^{i\theta}) = 0$ for each $e^{i\theta}$ in $S$, $S$ being both metrically dense and of second category on an arc $A$ of $C$, then $f(z) \equiv 0$.

2. To prove our theorem, we need two lemmas. The first one is very elementary in applying the notion of Baire categories. The second one is an extension of [4, Lemma 1].

Lemma. 1. Let $S$ be a set of second category in an interval $J$, then there exists a subinterval $I$ of $J$ such that $S \cap I$ is both dense and of second category in $I$.

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Proof. The existence of an interval in which $S$ is of second category and of an interval in which $S$ is dense are immediate consequences of the definition of "second category." We must prove the existence of an interval with both of these properties. By the remark following the proof of [5, Theorem 35, p. 201], there exists a non-empty closed subinterval $I'$ of $J$ such that the intersection of $S$ with any neighborhood $N$ of a point in $I'$ is of second category in $I'$. Thus a subinterval $I$ of $I'$ in which $S$ is dense has all the properties required.

Applying Lemma 1, it is easy to see that we can modify the proof of [4, Lemma 1, pp. 170–171] and obtain

**Lemma 2.** Suppose that $f(z)$ is a meromorphic function in $D$ and that for some fixed $\phi$, $-\pi/2 < \phi < \pi/2$, and some complex number $\beta$, finite or infinite, there exists a set $S$ of second category on $C$ and such that

$$\beta \in \bigcup_{e^{i\theta} \in S} C_{\rho(\theta)}(f, e^{i\theta}).$$

Then there exists an arc $A$ on $C$ such that

1. Either $f(z)$ or $1/(f(z) - \beta)$ is uniformly bounded in a relative neighborhood of $A$ in $D$ according as $\beta = \infty$ or $\beta \neq \infty$,
2. $A \cap S$ is dense in $A$, and
3. $A \cap S$ is of second category in $A$.

3. We now proceed to prove our theorem. Since $\lim_{r \to 1} \mu(r) = 0$ and $|f(re^{i\theta})| = o(\mu(r))$ for each $e^{i\theta}$ in $S$, we have

$$\infty \in \bigcup_{e^{i\theta} \in S} C_{\rho(\theta)}(f, e^{i\theta}).$$

Note that $S$ is of second category. By Lemma 2, there exists an arc $A$ of $C$ such that $f(z)$ is uniformly bounded in a relative neighborhood of $A$ in $D$ and that $A \cap S$ is of second category in $A$.

On the other hand, by a theorem [2, p. 6] of Barth and Schneider, there exists a nonconstant holomorphic function $g(z)$ in $D$ such that

$$\max_{0 \leq \theta < 2\pi} |g(re^{i\theta})| < 1/\mu(r) \quad \text{and} \quad \lim_{r \to 1} g(re^{i\theta}) = 0$$

for each $e^{i\theta}$ in $T$, where $T$ is a subset of $C$ of measure $2\pi$. Now, consider the function $h(z) = f(z)g(z)$. Note that

$$\lim_{r \to 1} h(re^{i\theta}) = 0$$

for each $e^{i\theta}$ in $(S \cup T) \cap A$. Since $(S \cup T) \cap A$ is both metrically dense and of second category in $A$, $h(z)$ satisfies the hypotheses of [4, Theorem 1] and we have $h(z) \equiv 0$. Hence $f(z) \equiv 0$. 

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4. Gauthier (in a written communication to the author) has kindly pointed out that our theorem is sharp in the following sense:

**Theorem B.** Let $\mu$ be as in Theorem A and $S$ be any set of first category on $C$. Then there exists a nonconstant holomorphic function $f(z)$ in $D$ such that $|f(re^{i\theta})| = o(\mu(r))$ for each $e^{i\theta}$ in $S$.

**Proof.** Since $S$ is of first category, $S = \bigcup_{n=1}^{\infty} S_n$, where $S_n$, for each $n = 1, 2, \ldots$, is nowhere dense on $C$. Thus $S' = \bigcup_{n=1}^{\infty} S_n$ is also of first category on $C$ and $S \subseteq S'$. Consider the set

$$E' = \bigcup_{n=1}^{\infty} \{re^{i\theta} : e^{i\theta} \subseteq S_n, 1 - 1/2n \leq r < 1\}.$$ 

Choose $\theta_0$ such that $e^{i\theta_0} \subseteq S'$ and let $E = E' \cup \rho(\theta_0)$. Note that $E$ is relatively closed and nowhere dense in $D$. We define

$$\phi(z) = \begin{cases} 0, & z \in E', \\ 1, & z \in \rho(\theta_0). \end{cases}$$

$\phi(z)$ is continuous on $E$. Hence by a theorem of Arakeljan (see Arakeljan [1]), there exists a holomorphic function $f(z)$ in $D$ such that $|f(z) - \phi(z)| \leq \mu(|z|)(1 - |z|)$ for each $z$ in $E$. It follows that $|f(re^{i\theta})| = o(\mu(r))$ for each $e^{i\theta}$ in $S'$ (note that $S \subseteq S'$), and $f(z) \neq 0$ because $f(z)$ is near to one on $\rho(\theta_0)$.

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**References**


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