COMPACTNESS IN L_∞ SPACES

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Conditions for compactness in $L_\infty(S, \Sigma, \mu)$ are known when $\mu(S)$ is finite [**1**, p. 297]. The purpose of this note is to state a compactness criterion which does not depend on $\mu(S)$. The proof uses the standard diagonal procedure. It should be noted that the criterion is not a necessary one unless the underlying space S is σ-finite (see remark at the end).

Preliminaries. We assume that (S, Σ, μ) is a positive measure space [**1**, p. 126]. The triple (S, Σ, μ) is said to be *σ-finite* whenever there is a sequence of sets in Σ of finite μ-measure whose union is S. A *finite decomposition* (*of* S) is a finite collection of pairwise disjoint sets in Σ whose union is S. A set $N \in \Sigma$ is called *locally μ-null* if $\mu(E \cap N) = 0$ for all $E \in \Sigma$ such that $\mu(E) < \infty$. Let $\mathfrak{N}$ denote the family of all locally μ-null sets. A complex-valued, μ-measurable function f on S [**1**, p. 106] is said to be *μ-essentially bounded* if for some real number $a \geqq 0$ we have $\{s : |f(s)| > a\} \in \mathfrak{N}$. Let

$$\|f\|_\infty = \inf\{a : \{s : |f(s)| > a\} \in \mathfrak{N}\}.$$

We denote by $L^0_\infty(S, \Sigma, \mu)$ the space of all complex-valued, μ-essentially bounded, μ-measurable functions on S. $L^0_\infty(S, \Sigma, \mu)$ is partitioned according to the equivalence relation: *f is equivalent to g* if and only if $\|f-g\|_\infty = 0$. Denote the class of all functions equivalent to f by $[f]$ and let $L_\infty(S, \Sigma, \mu)$ denote the space of all such equivalence classes $[f] \subseteq L^0_\infty(S, \Sigma, \mu)$. With the induced norm $\|[f]\|_\infty = \|f\|_\infty$, $L_\infty(S, \Sigma, \mu)$ is a Banach space [**2**, p. 347].

Definition 1. For any $f \in L^0_\infty(S, \Sigma, \mu)$ and nonempty set $E \in \Sigma$ we define the *μ-essential height* of f on E by the seminorm

$$|f|_E = \inf\{\sup\{|f(s) - f(t)| : s, t \in E - N \cap E\} : N \in \mathfrak{N}\}.$$

For convenience, we set $|f|_E = 0$ when E is the empty set.

Lemma. *Suppose (S, Σ, μ) is σ-finite. For $f \in L^0_\infty(S, \Sigma, \mu)$ and any $\epsilon > 0$ there corresponds a finite decomposition $\{E_k : k = 1, \cdots, n\}$ such that*

$$\max\{|f|_{E_k} : k = 1, \cdots, n\} < \epsilon.$$

Proof. We first show this when f is real-valued. Given $\epsilon > 0$ we construct a decomposition of S in the following way: For integers k

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which satisfy $\epsilon - \|f\|_\infty < k\epsilon < \|f\|_\infty$, set

$$G_k^* = \{s : (k-1)\epsilon < f(s) \leqq k\epsilon\}.$$

For the integer n which satisfies $(n-1)\epsilon \leqq \|f\|_\infty < n\epsilon$, set

$$G_n^* = \{s : (n-1)\epsilon < f(s) \leqq \|f\|_\infty\},$$

$$G_{-(n-1)}^* = \{s : -\|f\|_\infty \leqq f(s) \leqq -(n-1)\epsilon\},$$

and

$$G_{-n}^* = \{s : |f(s)| > \|f\|_\infty\}.$$

From [2, p. 347] we have $G_{-n}^* \in \mathfrak{N}$. Let (S, Σ^*, μ) be the Lebesgue extension of (S, Σ, μ) [1, p. 143]. From the σ-finite property and [1, p. 148], for each k, $G_k^* \in \Sigma^*$. By definition [1, p. 142], for some $G_k' \in \Sigma$ and $N_k' \in \mathfrak{N}$, we have $G_k^* = M_k \cup G_k'$ for some subset M_k of N_k'. Let

$$G_{-n} = G_{-n}' \cup \left(\bigcup_{k=-n}^{n} N_k' \right) \quad \text{and} \quad G_k = G_k' - G_k' \cap G_{-n}.$$

By construction, $\{G_k\}$ is pairwise disjoint and $\bigcup_{k=-n}^{n} G_k = S$. Thus, $\{G_k\}$ is a finite decomposition such that $|f|_{G_k} < \epsilon$ for each k. For a complex-valued function, we need only consider the intersections from the finite decompositions for the real and imaginary parts derived as above.

The following definition generalizes the notion of uniform equicontinuity [3, p. 239] quite naturally to families of functions in $L_\infty^0(S, \Sigma, \mu)$.

DEFINITION 2. A family of complex-valued μ-measurable functions, $\{f\}$, is *quasi-uniformly equi-μ-measurable* (abbreviated *que μ-measurable*) if and only if to each $\epsilon > 0$ there corresponds a finite decomposition $\{E_k : k = 1, \cdots, n\}$ such that on each E_k, the μ-essential height of every $f \in \{f\}$ is not greater than ϵ.

A *representative set* of $K \subseteq L_\infty(S, \Sigma, \mu)$ is any set K_r in $L_\infty^0(S, \Sigma, \mu)$ which consists of one and only one function from each equivalence class in K. It is easy to verify that whenever $K \subseteq L_\infty(S, \Sigma, \mu)$ has one representative set which is que μ-measurable then every representative set of K is que μ-measurable. Moreover, the existence of a representative set is assured by the Axiom of Choice. Thus, we may formulate a nontrivial definition of que μ-measurability in $L_\infty(S, \Sigma, \mu)$ without ambiguity.

DEFINITION 3. A subset K of $L_\infty(S, \Sigma, \mu)$ is said to be *que μ-measurable* if and only if every representative set of K is que μ-measurable.

THEOREM. *A bounded set K in $L_\infty(S, \Sigma, \mu)$ is conditionally compact if K is que μ-measurable. The converse holds when (S, Σ, μ) is σ-finite.*

PROOF. (*Necessity*) Given $\epsilon>0$ there are elements $[f_1], \cdots, [f_m]$ in K such that for any $[f]\in K$,

$$\min\{\|[f]-[f_j]\|_\infty : j = 1, \cdots, m\} \leqq \epsilon/3.$$

Let K_r denote any representative set for K and denote by $f_1, \cdots, f_m$ the representatives in K_r corresponding to $[f_1], \cdots, [f_m]$, respectively. Using the above lemma, it is not difficult to show that any finite set in $L^0_\infty(S, \Sigma, \mu)$ is que μ-measurable. Thus, there is a finite decomposition $\{E_k : k=1, \cdots, n\}$ such that $|f_j|_{E_k} \leqq \epsilon/3$ for $j=1, \cdots, m$ and $k=1, \cdots, n$. For any $f\in K_r$ and $s, t\in E_k$ we have

$$|f(s)-f(t)| \leqq |f(s)-f_j(s)| + |f_j(s)-f_j(t)| + |f_j(t)-f(t)|.$$

Thus, $|f|_{E_k} \leqq 2|f-f_j|_{E_k}+\epsilon/3$. Therefore, taking the minimum of both sides with respect to j we have $|f|_{E_k}\leqq\epsilon$. Hence, K_r is que μ-measurable.

(*Sufficiency*) Since K is bounded there is a constant M such that $\sup\{\|[f]\|_\infty : [f]\in K\}\leqq M$. Select a representative set K_r for K in which every function f satisfies $|f(s)|\leqq M$ for all $s\in S$. For any sequence $\{[f_n]\}$ in K let $\{f_n\}$ be its representative sequence in K_r. To each $\epsilon_k=1/k$ there corresponds a finite decomposition $\{E_{k,j} : j=1, \cdots, l_k\}$ such that

$$\max\{\sup\{|f_n|_{E_{k,j}} : f_n \in \{f_n\}\} : j = 1, \cdots, l_k\} \leqq \epsilon_k.$$

For each $E_{k,j}$ there is $F_{k,j}\subseteq E_{k,j}$ such that $E_{k,j}-F_{k,j}\in\mathfrak{N}$ and

$$\sup\{|f_n(s)-f_n(t)| : s, t \in F_{k,j} \text{ and } f_n \in \{f_n\}\} \leqq \epsilon_k.$$

This follows since a countable union of locally μ-null sets is locally μ-null. From each $F_{k,j}$ we select any element $s_{k,j}$. For a fixed $s_{k,j}$ the set $\{f_n(s_{k,j}) : f_n\in\{f_n\}\}$ is a bounded set of complex numbers. Thus, by the Bolzano-Weierstrass theorem, for $k=1$ there is a subsequence $\{f_{1_n}\}$ of $\{f_n\}$ which converges uniformly at the points $\{s_{1,j} : j=1, \cdots, l_1\}$. For $k=2$, there is a subsequence $\{f_{2_n}\}$ of $\{f_{1_n}\}$ which converges uniformly at the points $\{s_{1,i}\}\cup\{s_{2,j}\}$. Continuing in this way, by the diagonal method we obtain a subsequence $\{f_{n_n}\}$ which converges at each point of the set $\{s_{k,j} : k=1, \cdots ; j=1, \cdots, l_k\}$ and the convergence is uniform on any finite subset.

Given $\epsilon>0$, choose p such that $\epsilon_p\leqq\epsilon/3$. Since $\{f_{n_n}\}$ converges uniformly on $\{s_{p,j} : j=1, \cdots, l_p\}$, there is some integer N such that if $n, m\geqq N$ we have

$$\max\{ |f_{n_n}(s_{p,j}) - f_{m_m}(s_{p,j})| : j = 1, \cdots, l_p\} \leqq \epsilon/3.$$

For any $s \in F_{p,j}$ we have

$$\begin{aligned} |f_{n_n}(s) - f_{m_m}(s)| &\leqq |f_{n_n}(s) - f_{n_n}(s_{p,j})| + |f_{n_n}(s_{p,j}) - f_{m_m}(s_{p,j})| \\ &\quad + |f_{m_m}(s_{p,j}) - f_{m_m}(s)| \\ &\leqq 2\epsilon_p + \epsilon/3. \end{aligned}$$

Thus, $\|f_{n_n} - f_{m_m}\|_\infty \leqq \epsilon$ if $n, m \geqq N$. Therefore $\{[f_{n_n}]\}$ is a Cauchy sequence in K. Hence, K is conditionally compact.

REMARK. If (S, Σ, μ) is not σ-finite, the above criterion for compactness is not a necessary one as the following example illustrates: Let $S = [0, 1]$, Σ be the usual Borel field of sets in S, and μ be the counting measure which assigns to each set in Σ its cardinality. Select any nonmeasurable set $D \subset S$ and let $f(s) = 0$ for $s \in D$ and $f(s) = 1$ on $S - D$. From [1, p. 148], f is μ-measurable. However, f does not satisfy Definition 2 for any $\epsilon < 1$. Thus, the set $\{[f]\}$, although obviously compact in $L_\infty([0, 1], \Sigma, \mu)$, is not que μ-measurable.

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