ON $4/n=1/x+1/y+1/z$

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ABSTRACT. It is shown that the number of positive integers $n\leqq N$ for which $4/n=1/x+1/y+1/z$ is not solvable in positive integers, is less than a constant times $N/(\log N)^{7/4}$.

I. **Introduction.** Erdös has conjectured that the equation

$$4/n = 1/x + 1/y + 1/z \tag{I}$$

is solvable in positive integers for all integers $n\geqq 2$. This has not as yet been proved, but it is known that (I) is solvable for n less than some constant [1], [2], [5]. Using the methods found in these works and some fairly advanced analytic techniques, it can be shown that $S(N)\ll N/(\log N)^{\alpha}$ where α is a constant less than one, and $S(N)$ is the number of positive integers n less than N for which (I) is not solvable. In this paper it is shown that better estimates can be obtained using methods which are essentially elementary.

II. **Principal results.** By looking at the problem somewhat differently, we are able to obtain various conditions on n which imply the solvability of (I), and then apply sieve methods to obtain an upper bound for $S(N)$. The basic lemma needed is the following:

LEMMA 1. *$a/b=1/x+1/y$ if and only if there exist divisors d_1 and d_2 of b such that $a\mid(d_1+d_2)$ (a, b, x, y positive integers).*

A proof of a generalized form of the above lemma may be found in [6]. We will illustrate the method by considering primes modulo 8.

LEMMA 2. *Let p be a prime, then* (I) *is solvable if*:

(i) $p\equiv 7 \pmod 8$ *and* $n\equiv 0 \pmod p$ *or* $n+1\equiv 0 \pmod p$ *or* $n+2\equiv 0 \pmod p$ *or* $2n+1\equiv 0 \pmod p$ *or*

(ii) $p\equiv 3 \pmod 8$ *and* $n\equiv 0 \pmod p$ *or* $n+1\equiv 0 \pmod p$ *or*

(iii) $p\equiv 5 \pmod 8$ *and* $n\equiv 0 \pmod p$.

PROOF.

Case (i). Let $p=8t+7$, and $r=2(t+1)$. Then

$$4/n = 1/rn + p/2(t+1)n.$$

Received by the editors April 18, 1969 and, in revised form, November 21, 1969.

AMS Subject Classifications. Primary 1010; Secondary 1064.

Key Words and Phrases. Diophantine equation, divisors, residue classes, Selberg's sieve.

If $p|n$, the last fraction is reducible and (I) is solvable trivially. (Note: $1/x=1/(x+1)+1/x(x+1)$.) To obtain the other conditions, apply Lemma 1 to the following pairs of divisors of $2(t+1)n$: n and 1, n and 2, $2n$ and 1.

Case (ii). Let $p=4t+3$, and $r=t+1$. Then

$$4/n = 1/rn + p/(t+1)n.$$

If $p|n$, $p/(t+1)n$ is reducible; and if $p|n+1$ apply Lemma 1.

Case (iii). If $p\equiv 5 \pmod 8$, $p+1\equiv 6 \pmod 8$ which implies that $p+1$ has a prime divisor q such that $q=4r-1$. Then

$$4/n = 1/rn + q/rn.$$

If $p|n$, then $q|p+1$ and both p and 1 divide the denominator of the last fraction. Therefore we may apply Lemma 1 again.

THEOREM 1. $S(N)\ll N/(\log N)^{7/4}$.

PROOF. We apply Selberg's sieve to the positive integers $\leqq N$, where the sifting classes for a given prime are those given for n in the statement of Lemma 2. (Note that these residue classes are distinct.)

In particular, we apply Theorem 3, p. 213 of [3], which states that

$$S(N) \leqq NQ^{-1} + z \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{f(p)}\right)^{-2}$$

where

$$\begin{aligned}
\mathcal{P} &= \{p \mid p \text{ is a prime} \leqq N,\ p \not\equiv 1 \pmod 8,\ p \neq 2\},\\
\mathcal{A} &= \{n \mid n \text{ is a positive integer} \leqq N\},\\
f(p) &= p/4 \qquad \text{if } p \equiv 7 \pmod 8\\
&= p/2 \qquad \text{if } p \equiv 3 \pmod 8\\
&= p \qquad\ \ \text{if } p \equiv 5 \pmod 8,\\
\prod(\mathcal{P}) &= \prod_{p\in\mathcal{P}} p,\\
f(d) &= \prod_{p|d} f(p) \qquad \text{for } d \,\Big|\, \prod(\mathcal{P}),\\
\mathfrak{D} &= \{d \mid d \text{ divides } \textstyle\prod(\mathcal{P}),\ d \leqq z^{1/2}\},\\
z &= N^{2/3},\\
Q &= Q(\mathfrak{D}) = \sum_{d\in\mathfrak{D}} \frac{1}{g(d)},
\end{aligned}$$

and

$$g(d) = f(d) \prod_{p|d} \left(1 - \frac{1}{f(p)}\right),$$

provided $|R_d| \leqq d/f(d)$ where

$$\sum_{n \in \mathfrak{A};\, d|\sigma(n)} 1 = N/f(d) + R_d$$

and $\sigma(n) = \prod p_i$ where the product is over all primes p_i which are moduli of sifting classes containing n.

If $d = p_1 p_2 \cdots p_r$, then

$$\sum_{n \in \mathfrak{A};\, d|\sigma(n)} 1 = \text{number of } n \leqq N \text{ which are sifted by } p_1, p_2, \cdots, \text{ and } p_r,$$

$$= \text{number of } n \leqq N \text{ which satisfy a system of congruences:}$$

(II)

$$\begin{aligned} n &\equiv h_1 \pmod{p_1} \\ n &\equiv h_2 \pmod{p_2} \\ &\vdots \\ n &\equiv h_r \pmod{p_r} \end{aligned}$$

where h_i is any one of the $p_i/f(p_i)$ residue classes sifted by our sieve.

The system (II) is equivalent to a congruence

$$n \equiv H_j \pmod{d}$$

and there are $(p_1/f(p_1)) \cdots (p_r/f(p_r)) = d/f(d)$ such congruences. For each such congruence there are $(N/d + E_j)$ n which are $\leqq N$ and satisfy the congruence, and $|E_j| \leqq 1$. Therefore

$$\sum_{n \in \mathfrak{A};\, d|\sigma(n)} 1 = \sum_{j=1}^{d/f(d)} (N/d + E_j) = \frac{N}{f(d)} + R_d$$

where $|R_d| = \left|\sum_{j=1}^{d/f(d)} E_j\right| \leqq d/f(d)$.

To complete the proof of Theorem 1, we need only show that

$$NQ^{-1} + z \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{f(p)}\right)^{-2} \ll \frac{N}{(\log N)^{7/4}} \cdot$$

(III) $$Q = Q(\mathfrak{D}) = \sum_{d \in \mathfrak{D}} \frac{1}{g(d)} \geqq \sum_{d \in \mathfrak{D}} \frac{1}{f(d)}$$

$$\geqq \left(\sum_{d \leqq N^{1/9};\, d \in \mathfrak{D}_5} \frac{1}{d}\right)\left(\sum_{d \leqq N^{1/9};\, d \in \mathfrak{D}_3} \frac{2^{\Omega(d)}}{d}\right)\left(\sum_{d \leqq N^{1/9};\, d \in \mathfrak{D}_7} \frac{4^{\Omega(d)}}{d}\right)$$

where $\mathfrak{D}_j=\{d \mid d\in\mathfrak{D}$ and $p \mid d$ implies $p\equiv j \pmod 8\}$ and $\Omega(d)=$total number of primes dividing d. (Since d is square free, $\Omega(d)=\omega(d)$, the number of different primes dividing d; but it is convenient to use Ω rather than ω.)

Hence, we need estimates on sums of the form:

$$\sum \frac{b^{\Omega(n)}}{n}.$$

To facilitate these estimates, we assume until further notice that the only integers n we deal with have the property that if $p \mid n$ then $p>b$.

Let

$$T(y) = \sum_{n\leq y}' \frac{b^{\Omega(n)}}{n}$$

where $\sum'$ denotes a sum over square free numbers. Also, let

$$S_{l,x} = \{n \leqq x \mid l^2 \text{ is the largest square factor of } n\}.$$

Then

$$\begin{aligned}\sum_{n\leqq x}\frac{b^{\Omega(n)}}{n} &= \sum_{j=1}^{[\sqrt{x}]}\sum_{n\in S_{j,x}}\frac{b^{\Omega(n)}}{n} = \sum_{j=1}^{[\sqrt{x}]}\sum_{n\in S_{1,x/j^2}}\frac{b^{\Omega(j^2n)}}{j^2n}\\ &= \sum_{j=1}^{[\sqrt{x}]}\frac{b^{\Omega(j^2)}}{j^2}\sum_{n\in S_{1,x/j^2}}\frac{b^{\Omega(n)}}{n} \leqq \sum_{j=1}^{[\sqrt{x}]}\frac{b^{\Omega(j^2)}}{j^2}\sum_{n\in S_{1,x}}\frac{b^{\Omega(n)}}{n}\\ &= T(x)\sum_{j=1}^{[\sqrt{x}]}\frac{(b^2)^{\Omega(j)}}{j^2} \leqq T(x)\prod_{b<p\leqq x^{1/2}}\left(1+\frac{b^2}{p^2}+\frac{b^4}{p^4}+\cdots\right)\\ &\leqq T(x)\prod_{p>b}\left(1+c_1\frac{b^2}{p^2}\right) \leqq c_2T(x).\end{aligned}$$

(c_i will always denote an unspecified constant.) Hence,

$$\text{(IV)} \qquad \sum_{n\leqq x}'\frac{b^{\Omega(n)}}{n} \geqq c_3\sum_{n\leqq x}\frac{b^{\Omega(n)}}{n}.$$

Now

$$\sum_{n\leqq y}\frac{1}{n} \geqq c_4\left(\sum_{n\leqq y}\frac{1}{n}\right)\left(\sum_{j=0}^{\infty}\frac{1}{2^j}\right)\left(\sum_{j=0}^{\infty}\frac{1}{3^j}\right)\cdots\left(\sum_{j=0}^{\infty}\frac{1}{p_s^j}\right)$$

where $p_s \leqq b < p_{s+1}$. Therefore

(V) $$\sum_{n\leqq y}\frac{1}{n}\geqq c_4\sum_{m\leqq y}\frac{1}{m}\geqq c_5\log y$$

where the last sum is over *all* positive integers $\leqq y$.

Now

(VI) $$\sum_{n\leqq x}\frac{b^{\Omega(n)}}{n}\geqq\sum_{n\leqq x}\frac{A(n)}{n}=\left(\sum_{n\leqq x^{1/b}}\frac{1}{n}\right)^n$$

where $A(n)$ = number of ways n can be written as a product of b numbers, each less than $x^{1/b}$. That $b^{\Omega(n)}\geqq A(n)$ can be seen from the fact that we can assign each prime factor of n to any one of b factors. We get every possible factorization of n in this way, but may get some not counted in $A(n)$. Thus, by (IV), (V), and (VI) we obtain:

(VII) $$\sum_{n\leqq x}{}'\frac{b^{\Omega(n)}}{n}\geqq c_6(\log x)^b.$$

Since

$$\prod_{b<p\leqq x}\left(1-\frac{b}{p}\right)^{-1}<c_7\prod_{b<p\leqq x}\left(1-\frac{1}{p}\right)^{-b}\quad\text{[4, Satz 5.5]}$$

and

$$\prod_{b<p\leqq x}\left(1-\frac{1}{p}\right)^{-b}\leqq c_8(\log x)^b\quad\text{[4, Satz 4.1]}$$

by (VII):

(VIII) $$\sum_{n\leqq x}{}'\frac{b^{\Omega(n)}}{n}\geqq c_9\prod_{b<p\leqq x}\left(1-\frac{b}{p}\right)^{-1}.$$

Let L be any set of primes, q an element of L, $L'=L-\{q\}$, $M_L=\{m\mid m$ is a positive integer $\leqq M$, and $p\mid m$ implies $p\in L\}$, and $M_{L'}$ defined similarly. We now show

(IX) $$\prod_{p\in L}\left(1-\frac{b}{p}\right)^{-1}\leqq c_{10}\sum_{m\in M_L}{}'\frac{b^{\Omega(m)}}{m}$$

implies

$$\prod_{p\in L'}\left(1-\frac{b}{p}\right)^{-1}\leqq c_{10}\sum_{m\in M_{L'}}{}'\frac{b^{\Omega(m)}}{m}.$$

If

$$\prod_{p\in L}\left(1-\frac{b}{p}\right)^{-1} \leqq c_{10} \sum_{m\in M_L}{}' \frac{b^{\Omega(m)}}{m}$$

then

$$\begin{aligned}\prod_{p\in L'}\left(1-\frac{b}{p}\right)^{-1} &= \prod_{p\in L}\left(1-\frac{b}{p}\right)^{-1}\left(1-\frac{b}{q}\right)\\ &\leqq c_{10}\sum_{m\in M_L}{}' \frac{b^{\Omega(m)}}{m}\left(1-\frac{b}{q}\right)\\ &= c_{10}\left(\sum_{m\in M_L}{}' \frac{b^{\Omega(m)}}{m} - \sum_{m\in M_L}{}' \frac{bb^{\Omega(m)}}{qm}\right)\\ &\leqq c_{10}\left(\sum_{m\in M_L}{}' \frac{b^{\Omega(m)}}{m} - \sum_{m\in M_L;\, q\mid m}{}' \frac{b^{\Omega(m)}}{m}\right)\\ &= c_{10}\sum_{m\in M_{L'}}{}' \frac{b^{\Omega(m)}}{m}.\end{aligned}$$

By (VIII) and repeated use of (IX) we have

$$\text{(X)}\qquad \sum_{d\leqq N^{1/9};\, d\in\mathfrak{D}_j} \frac{b_j^{\Omega(d)}}{d} \geqq c_9 \prod_{p\leqq N^{1/9};\, p\equiv j\,(\mathrm{mod}\ 8)}\left(1-\frac{b_j}{p}\right)^{-1}$$

where $b_3=2$, $b_5=1$ and $b_7=4$. (The condition $p>b_j$ is vacuous here.) Now

$$\begin{aligned}\log\left(\prod_{p\leqq N^{1/9};\, p\equiv j\,(\mathrm{mod}\ 8)}\left(1-\frac{b_j}{p}\right)^{-1}\right) &= -\sum_{p\leqq N^{1/9};\, p\equiv j\,(\mathrm{mod}\ 8)} \log\left(1-\frac{b_j}{p}\right)\\ &\geqq \sum_{p\leqq N^{1/9};\, p\equiv j\,(\mathrm{mod}\ 8)} \frac{b_j}{p} \geqq \frac{b_j \log\log N^{1/9}}{\phi(8)} + c_{11}\\ &\geqq \frac{b_j}{4}\log\log N + c_{12}\end{aligned}$$

and therefore

$$\begin{aligned}\text{(XI)}\qquad \prod_{p\leqq N^{1/9};\, p\equiv j\,(\mathrm{mod}\ 8)}\left(1-\frac{b_j}{p}\right)^{-1} &\geqq \exp\left(\frac{b_j}{4}\log\log N + c_{12}\right)\\ &= c_{13}(\log N)^{b_j/4}.\end{aligned}$$

Hence, by (III), (X), and (XI)

$$Q(\mathfrak{D}) \geqq c_{14}(\log N)^{(b_3+b_5+b_7)/4} = c_{14}(\log N)^{7/4}.$$

Note that if $d \in \mathfrak{D}_j$ and $p \mid d$, then $p > b_j$. Finally,

$$\prod_{p \leq \mathcal{P}} \left(1 - \frac{1}{f(p)}\right)^{-2} \leqq \prod_{p \leq N} \left(1 - \frac{4}{p}\right)^{-2} \leqq c_{15}(\log N)^8$$

by arguments essentially the same as used above. Therefore

$$z \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{f(p)}\right)^{-2} \leqq c_{15} N^{2/3} (\log N)^8$$

and so

$$\begin{aligned} S(N) &\leqq N \frac{c_{16}}{(\log N)^{7/4}} + c_{15} N^{2/3} (\log N)^8 \\ &\leqq c_{17} \frac{N}{(\log N)^{7/4}} . \end{aligned}$$

This completes the proof of Theorem 1.

III. **Concluding remarks.** By considering the primes in various residue classes modulo 16, the results of Theorem 1 can be improved to

$$S(N) \ll N/(\log N)^2.$$

The exponent of log N may be improved to $9/4-\epsilon$ by considering primes modulo 2^k for arbitrary k (ϵ any small positive number).

The results are still a long way from the conjecture that $S(N)=0$, or even from $S(N) \ll N^{1-\epsilon}$, which would be quite desirable to prove.

References

1. Alexander Aigner, *Brüche als Summe von Stammbrüchen*, J. Reine Angew. Math. **214/215** (1964), 174–179. MR **28** #3969.

2. L. Bernstein, *Zur Lösung der diophantischen Gleichung $m/n=1/x+1/y+1/z$, insbesondere im Fall $m=4$*, J. Reine Angew. Math. **211** (1962), 1–10. MR **26** #77.

3. H. Halberstam and K. F. Roth, *Sequences*, Vol. 1, Clarendon Press, Oxford, 1966. MR **35** #1565.

4. Karl Prachar, *Primzahlverteilung*, Springer-Verlag, Berlin and New York, 1957. MR **19**, 393.

5. B. M. Stewart, *Theory of numbers*, 2nd ed., Macmillan, New York, 1964. MR **37** #6232.

6. B. M. Stewart and W. A. Webb, *Sums of fractions with bounded numerators*, Canad. J. Math. **18** (1966), 999–1003. MR **33** #7297.

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