INEQUALITIES FOR A SYMMETRIC ELLIPTIC INTEGRAL[1]

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ABSTRACT. Inequalities are found for an incomplete elliptic integral of the first kind which represents the reciprocal of the capacity of an ellipsoid with semiaxes x, y, z. One sequence of symmetric algebraic functions of x, y, z converges to the value of the integral from below and two from above. Among the elements of these sequences are upper and lower approximations due to Pólya and Szegö.

1. Introduction and summary. Let x, y, z be positive numbers and define

$$R = \frac{1}{2}\int_0^\infty [(t+x^2)(t+y^2)(t+z^2)]^{-1/2}dt. \tag{1.1}$$

The electric capacity of a conducting ellipsoid with semiaxes x, y, z is $1/R$ [1]. In terms of Legendre's elliptic integral $F(\phi, k)$ and the symmetric elliptic integral $R_F(x, y, z)$ [2], we have

$$R = R_F(x^2, y^2, z^2) = (z^2 - x^2)^{-1/2}F\left[\cos^{-1}\frac{x}{z}, \left(\frac{z^2-y^2}{z^2-x^2}\right)^{1/2}\right]. \tag{1.2}$$

It is useful for numerical and analytical purposes to approximate R by an algebraic function, preferably one which, like R itself, is symmetric and homogeneous of degree -1 in x, y, z and has the value unity if $x=y=z=1$. Some possible candidates are

$$\begin{aligned}
&\alpha = 3/\sum yz/x, \qquad \beta = (3/\sum x^2)^{1/2}, \qquad \gamma = (3/\sum xy)^{1/2},\\
&\delta = 3/\sum (xy)^{1/2}, \qquad \epsilon = (xyz)^{-1/3}, \qquad \zeta = \tfrac{1}{3}\sum 1/x,\\
&\eta = (\tfrac{1}{3}\sum 1/x^2)^{1/2}, \qquad \theta = \tfrac{1}{3}\sum x/yz, \qquad \alpha_1 = 3/\sum x,\\
&\epsilon_1 = \frac{2}{[(x+y)(y+z)(z+x)]^{1/3}}, \qquad \eta_1 = \left(\frac{4}{3}\sum\frac{1}{(x+y)(x+z)}\right)^{1/2},\\
&\theta_1 = \frac{2}{3}\sum\frac{1}{x+y}, \qquad \alpha_2 = \frac{6}{\sum[(x+y)(x+z)]^{1/2}},
\end{aligned} \tag{1.3}$$

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where $\sum$ denotes a summation over the three cyclic permutations of x, y, z.

We henceforth exclude the case $x=y=z$. In 1917 Pólya [3] stated the inequality

$$\alpha < R < \theta \tag{1.4}$$

in a problem. The solution given by Szegö [4] showed further that

$$\alpha < \beta < R < \epsilon < \eta < \theta. \tag{1.5}$$

Indeed, $R<\epsilon$ is a special case of Poincaré's theorem [5] that a sphere has a smaller capacity than any other conductor of the same volume. In 1945 Pólya and Szegö [1] proved a still sharper inequality,

$$\alpha_1 < R < \delta. \tag{1.6}$$

It has recently been shown [6] by W. H. Greiman that

$$\alpha_1 < R < \epsilon_1 \tag{1.7}$$

and by Carlson [7] that $\epsilon_1<\delta$.

Let $\alpha_n, \cdots, \theta_n$ denote the result of replacing x, y, z in the expressions for $\alpha, \cdots, \theta$ by x_n, y_n, z_n, where

$$\begin{gathered} x_0 = x, \qquad y_0 = y, \qquad z_0 = z, \\ x_{n+1} = \left(\frac{x_n + y_n}{2} \frac{x_n + z_n}{2}\right)^{1/2}, \qquad y_{n+1} = \left(\frac{y_n + z_n}{2} \frac{y_n + x_n}{2}\right)^{1/2}, \\ z_{n+1} = \left(\frac{z_n + x_n}{2} \frac{z_n + y_n}{2}\right)^{1/2}, \qquad n = 0, 1, 2, \cdots. \end{gathered} \tag{1.8}$$

Thus $\alpha_1, \epsilon_1, \eta_1, \theta_1, \alpha_2$ have the values given in (1.3). In the present note we prove that if $n \geqq 2$

$$\alpha < \beta < \alpha_1 < \beta_1 < \cdots < \alpha_n < \beta_n < R, \tag{1.9}$$

$$R < \zeta_n < \eta_n < \theta_n < \cdots < \zeta_1 < \eta_1 < \theta_1 < \zeta < \eta < \theta, \tag{1.10}$$

$$\begin{aligned} R &< \gamma_n < \delta_n < \epsilon_n < \zeta_n < \eta_n < \cdots \\ &< \gamma_1 < \delta_1 < \epsilon_1 < \zeta_1 < \eta_1 < \gamma < \delta < \epsilon < \zeta < \eta. \end{aligned} \tag{1.11}$$

These inequalities contain all the results quoted earlier, and β_n, ζ_n, and γ_n approach R as $n \to \infty$. Two sequences of upper bounds are given because θ_{n+1} is not comparable with γ_n, δ_n, or ϵ_n. The inequalities tend to be sharp when the ratios of x, y, and z are close to unity.

One reasonable compromise between accuracy and algebraic simplicity is $\alpha_2 < R < \epsilon_1$, i.e.

$$\frac{6}{\sum[(x+y)(x+z)]^{1/2}} < \frac{1}{2}\int_0^\infty [(t+x^2)(t+y^2)(t+z^2)]^{-1/2}dt < \frac{2}{[(x+y)(y+z)(z+x)]^{1/3}}. \tag{1.12}$$

For example, in the case $x=1$, $y=2$, $z=3$, $R=0.5086446\cdots$, Equations (1.5), (1.6) and (1.12) yield

$$0.37 < 0.46 < R < 0.55 < 0.67 < 0.78, \tag{1.5'}$$

$$0.500 < R < 0.536, \tag{1.6'}$$

$$0.5081 < R < 0.5109. \tag{1.12'}$$

Inequalities for inverse circular and hyperbolic functions follow from

$$\begin{aligned} R_F(x^2, 1, 1) &= (1-x^2)^{-1/2}\cos^{-1} x, & 0 \leqq x < 1, \\ R_F(x^2, 1, 1) &= (x^2-1)^{-1/2}\cosh^{-1} x, & x > 1. \end{aligned} \tag{1.13}$$

For example (1.12) implies

$$\frac{6(1-x)^{1/2}}{2\sqrt{2}+(1+x)^{1/2}} < \cos^{-1} x < \frac{2^{2/3}(1-x)^{1/2}}{(1+x)^{1/6}}, \qquad 0 \leqq x < 1. \tag{1.14}$$

The ratio of the third member to the first increases monotonically from 1 at $x=1$ to 1.013 at $x=0$.

If exactly one of the numbers x, y, z is zero, then ϵ, ζ, η, and θ are infinite but the inequalities between finite quantities remain valid. However, R is then a complete elliptic integral for which inequalities preferable to (1.12) can readily be obtained from Gauss' algorithm of the arithmetic-geometric mean [2, Equation (5.3)], e.g.

$$\left(\frac{2}{x^{1/2}+y^{1/2}}\right)^2 < \frac{2}{\pi} R_F(x^2, y^2, 0) < (xy)^{-1/4}\left(\frac{2}{x+y}\right)^{1/2}, \qquad (x>0,\ y>0,\ x \neq y). \tag{1.15}$$

Some inequalities for integrals more general than (1.1), including the capacity and surface area of ellipsoids in n dimensions, are given in [**8**]. Some unsymmetrical nonalgebraic upper and lower bounds for (1.1) can be deduced from [**9**].

2. **Lower bounds.** We shall sharpen an inequality such as (1.5) by successive applications of the duplication theorem for elliptic integrals. This theorem has been used for iterative computation of R [**10**], but the quantities encountered in the iteration are not completely

symmetric in x, y, z. Besides the duplication theorem we shall use only two elementary results. First, the harmonic mean, the geometric mean, the arithmetic mean, and the root-mean-square form an increasing sequence. Second, Maclaurin's inequality for elementary symmetric functions states that

$$(abc)^{1/3} < \left(\frac{ab+bc+ca}{3}\right)^{1/2} < \frac{a+b+c}{3} \tag{2.1}$$

provided the positive numbers a, b, c are not all equal. With $a=x^2$, $b=y^2$, $c=z^2$, and $t>0$, (2.1) implies

$$(t+(xyz)^{2/3})^3 < (t+x^2)(t+y^2)(t+z^2) < \left(t+\frac{x^2+y^2+z^2}{3}\right)^3,$$

as observed in [4]. Substituting in (1.1) we have

$$\beta < R < \epsilon, \tag{2.2}$$

a result which follows also from [8, Theorem 2].

Continuing to exclude the case $x=y=z$, we note that $\alpha<\beta$ is implied by the identity

$$\frac{9}{\alpha^2}-\frac{9}{\beta^2} = \sum\left(\frac{yz}{x}\right)^2 + 2\sum x^2 - 3\sum x^2 = \frac{1}{2}\sum x^2\left(\frac{y}{z}-\frac{z}{y}\right)^2. \tag{2.3}$$

Another proof, given in [4], consists in applying (2.1) to $(a, b, c) = (yz/x, zx/y, xy/z)$. Furthermore the inequality of the arithmetic mean and the root-mean-square implies $\beta<\alpha_1$. Since $\alpha<\beta<\alpha_1$ implies $\alpha_n<\beta_n<\alpha_{n+1}$ by substitution of x_n, y_n, z_n for x, y, z, we have

$$\alpha < \beta < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots. \tag{2.4}$$

The duplication theorem for elliptic integrals [10], [6] states that $R_F(x_n^2, y_n^2, z_n^2)$ is independent of n. Taken with (2.2) this implies

$$\beta_n < R < \epsilon_n, \qquad n = 0, 1, 2, \cdots. \tag{2.5}$$

From (2.4) and (2.5) we deduce (1.9). Moreover, x_n, y_n, and z_n approach the common limit $1/R$ as $n\to\infty$ [10], and it follows that β_n and ϵ_n approach R.

3. **Upper bounds.** We observe first that $\gamma<\delta<\epsilon<\zeta<\eta<\theta$. The inequality of the arithmetic mean and the root-mean-square shows that $\gamma<\delta$ and $\zeta<\eta$, the inequality of the geometric and arithmetic means shows that $\delta<\epsilon$, and the inequality of the harmonic and geometric means shows that $\epsilon<\zeta$. We deduce $\eta<\theta$ from the identity

$$
(3.1)\qquad \begin{aligned} 9\theta^2 - 9\eta^2 &= \sum \frac{x^2}{y^2z^2} + 2\sum \frac{1}{x^2} - 3\sum \frac{1}{x^2} \\ &= \frac{1}{2}\sum \frac{1}{x^2}\left(\frac{y}{z} - \frac{z}{y}\right)^2 \end{aligned}
$$

or alternatively, as in [4], by applying (2.1) to $(a, b, c) = (x/yz, y/zx, z/xy)$.

Now θ_1 is not comparable with γ, δ, or ϵ because $\theta_1 < \gamma < \delta < \epsilon$ if $x \ll y = z$ whereas $\gamma < \delta < \epsilon < \theta_1$ if $x = y \ll z$. However, we may conclude that $\theta_1 < \zeta$ from the identity

$$
(3.2)\qquad 3\zeta - 3\theta_1 = \sum \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right) - \sum \frac{2}{x+y} = \frac{1}{2}\sum \frac{(x-y)^2}{xy(x+y)}
$$

or alternatively from Minkowski's inequality [11, p. 30] for the harmonic mean. Now $\theta_1 < \zeta < \eta < \theta$ implies $\theta_{n+1} < \zeta_n < \eta_n < \theta_n$ and hence

$$
(3.3)\qquad \cdots < \zeta_2 < \eta_2 < \theta_2 < \zeta_1 < \eta_1 < \theta_1 < \zeta < \eta < \theta.
$$

Since $R < \epsilon_n < \zeta_n$ by (2.5), we have proved (1.10).

To prove $\eta_1 < \gamma$ we use the inequality of the arithmetic and geometric means to show that

$$
(3.4)\qquad \left(\sum x\right)\left(\sum xy\right) > 3(xyz)^{1/3}\, 3(xyz)^{2/3} = 9xyz
$$

and hence

$$
(3.5)\qquad (x+y)(y+z)(z+x) = \left(\sum x\right)\left(\sum xy\right) - xyz > \frac{8}{9}\left(\sum x\right)\left(\sum xy\right).
$$

It follows that

$$
(3.6)\qquad \eta_1^2 = \frac{8\sum x}{3(x+y)(y+z)(z+x)} < \frac{3}{\sum xy} = \gamma^2.
$$

Now $\eta_1 < \gamma < \delta < \epsilon < \zeta < \eta$ implies $\eta_{n+1} < \gamma_n < \delta_n < \epsilon_n < \zeta_n < \eta_n$ and hence

$$
(3.7)\qquad \begin{aligned} \cdots &< \gamma_2 < \delta_2 < \epsilon_2 < \zeta_2 < \eta_2 < \gamma_1 < \delta_1 < \epsilon_1 < \zeta_1 < \eta_1 \\ &< \gamma < \delta < \epsilon < \zeta < \eta. \end{aligned}
$$

From (2.5) and (3.7) we deduce (1.11).

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