ON A TRANSFORMATION OF BILATERAL SERIES WITH APPLICATIONS

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ABSTRACT. This paper is devoted to the study of a simple transformation of bilateral series. Formulae for basic bilateral hypergeometric series and generalizations of theorems on mock theta functions are proved.

1. **Introduction.** The object of this paper is to study the following very elementary result.

TRANSFORMATION LEMMA. *Subject to suitable convergence conditions, if* $c_n = \sum_{m=0}^{\infty} a_{m+n} b_m$, *then*

$$\sum_{m=0}^{\infty} b_m \sum_{n=-\infty}^{\infty} a_n = \sum_{n=-\infty}^{\infty} c_n.$$

The proof of this result is quite simple

$$\sum_{m=0}^{\infty} b_m \sum_{n=-\infty}^{\infty} a_n = \sum_{m=0}^{\infty} b_m \sum_{n=-\infty}^{\infty} a_{n+m} = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} b_m a_{n+m} = \sum_{n=-\infty}^{\infty} c_n.$$

In the cases we shall treat all series will be absolutely convergent, in particular for $\sum_m \sum_n \alpha_{mn} = \sum_n \sum_m \alpha_{mn}$ it is sufficient to have both double series absolutely convergent.

This simple transformation has been used in one form or another by several authors [2], [3], [6], [11]. Our object here is to systematically explore several special cases of this result. In particular, it was hoped that it might be possible to sum the basic bilateral hypergeometric series ${}_2\psi_2$. Such a summation formula was originally sought by Bailey [4]; Agarwal and Verma [1] state that they were motivated to develop the theory of bi-basic hypergeometric series in the hope of finding such a formula. An incorrect summation of the ${}_2\psi_2$ is given in [10]; this may be seen by substituting $b \to bt$, $c \to ct$, $d \to dt$, $e \to et$ in [10, Equation (1)] and verifying that the series satisfies a first order q-difference equation in t which is inconsistent with a first order q-difference equation satisfied by the product. Unfor-

Received by the editors November 10, 1969.

AMS Subject Classifications. Primary 1048, 3320.

Key Words and Phrases. Basic bilateral hypergeometric series, Ramanujan, mock theta functions.

[1] Partially supported by National Science Foundation grant GP-8075.

tunately, the most we could obtain in this direction is enunciated in Theorem 4.

2. **Applications.** In this section we refer to the Transformation Lemma as TL. We utilize the notation

$$(A;q)_\infty = (A)_\infty = \prod_{j=0}^{\infty}(1-Aq^j), \quad (A;q)_n = (A)_n = (A)_\infty/(Aq^n)_\infty,$$

$${}_r\phi_s\begin{bmatrix} A_1, \cdots, A_r; t, q \\ B_1, \cdots, B_s \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(A_1)_n \cdots (A_r)_n t^n}{(B_1)_n \cdots (B_s)_n (q)_n},$$

$${}_r\psi_s\begin{bmatrix} A_1, \cdots, A_r; t, q \\ B_1, \cdots, B_s \end{bmatrix} = \sum_{n=-\infty}^{\infty} \frac{(A_1)_n \cdots (A_r)_n}{(B_1)_n \cdots (B_s)_n} t^n,$$

and

$$\prod\begin{bmatrix} A_1, \cdots, A_r; q \\ B_1, \cdots, B_s \end{bmatrix} = \prod_{i=1}^{r}(A_i)_\infty \Big/ \prod_{j=1}^{s}(B_j)_\infty.$$

THEOREM 1.

$${}_1\psi_1\begin{bmatrix} \alpha; t, q \\ \gamma \end{bmatrix} = \prod\begin{bmatrix} \gamma/\alpha\beta t, & \gamma/\alpha, & \gamma/\beta; q \\ \gamma/\alpha t, & \gamma, & \gamma/\alpha\beta \end{bmatrix} {}_1\psi_1\begin{bmatrix} \alpha; t, q \\ \gamma/\beta \end{bmatrix}.$$

PROOF. In the TL, take

$$a_n = \frac{(\alpha)_n t^n}{(\gamma)_n}, \qquad b_n = \frac{(\beta)_n(\gamma/\alpha\beta t)^n}{(q)_n},$$

$$c_n = \prod\begin{bmatrix} \gamma/\alpha, & \gamma/\beta; q \\ \gamma, & \gamma/\alpha\beta \end{bmatrix} \frac{(\alpha)_n t^n}{(\gamma/\beta)_n}.$$

Then $c_n = \sum_{m=0}^{\infty} a_{m+n}\, b_m$ is merely a restatement of the q-analog of Gauss's theorem [**9**, p. 97, Equation (3.3.2.5)]. Hence by the TL, we obtain Theorem 1.

COROLLARY (RAMANUJAN [**5**, p. 194, Equation (1.3)]).

$${}_1\psi_1\begin{bmatrix} \alpha; t, q \\ \gamma \end{bmatrix} = \prod\begin{bmatrix} q/\alpha t, & \gamma/\alpha, & q, & \alpha t; q \\ \gamma/\alpha t, & \gamma, & q/\alpha, & t; \end{bmatrix}.$$

PROOF. Put $\beta = \gamma/q$, and utilize

$${}_1\psi_1\begin{bmatrix} \alpha; t, q \\ q \end{bmatrix} = {}_1\phi_0[\alpha; t, q] = (\alpha t)_\infty/(t)_\infty,$$

by [**9**, p. 92, Equation (3.2.2.11)].

THEOREM 2. *If*

$$h(z) = (-zq^2; q^4)_\infty \left\{ \sum_{n=0}^{\infty} q^{4n^2} z^{2n}/(-zq^2; q^4)_n + \sum_{n=1}^{\infty} q^{2n^2} z^{-n}(-z^{-1}q^2; q^4)_n \right\},$$

then

$$h(z) + zqh(zq^2) = \frac{(q^2; q^2)_\infty(-zq; q^2)_\infty(-z^{-1}q; q^2)_\infty}{(-q^2; q^2)_\infty(q; q^5)_\infty(q^4; q^5)_\infty}.$$

PROOF. Let $b_m = q^{m^2}/(q^4; q^4)_m$, $a_m = q^{m^2}z^m$, $c_n = q^{n^2}z^n(-zq^{2n+2}; q^4)_\infty$. Then $\sum_{m=0}^{\infty} a_{m+n} b_m = c_n$ is merely an elementary identity of Euler [**9**, p. 93, Equation (3.2.2.18)], while

$$\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q^4; q^4)_m} = \{(q; q^5)_\infty(q^4; q^5)_\infty(-q^2; q^2)_\infty\}^{-1},$$

by [**8**, p. 154, Equation (20)], and

$$\sum_{m=-\infty}^{\infty} a_m = (q^2; q^2)_\infty(-zq; q^2)_\infty(-z^{-1}q; q^2)_\infty$$

is merely Jacobi's identity [**9**, p. 86, Equation (3.1.12)] which is actually a limiting case of the corollary of Theorem 1. Finally by breaking $\sum_{n=-\infty}^{\infty} c_n$ into even and odd terms, we obtain

$$h(z) + zqh(zq^2) = \sum_{n=-\infty}^{\infty} c_n$$

which yields Theorem 2.

Theorem 2 implies a result concerning Ramanujan's mock theta functions, since

$$h(1) = (-q^3; q^4)_\infty(F_0(q^2) + \varphi_0(q^2) - 1),$$
$$h(q^2) = (-q^4; q^4)_\infty(f_1(q^4) + 2\psi_1(q^4)),$$

where $F_0(q)$, $\varphi_0(q)$, $f_1(q)$, and $\psi_1(q)$ are mock theta functions [**11**, p. 278]. A result similar to Theorem 2 may be obtained by utilizing $b_m = q^{m^2+2m}/(q^4; q^4)_m$, $a_m = q^{m^2}z^m$; $\sum b_m$ now becomes an infinite product by [**8**, p. 153, Equation (16)].

THEOREM 3. *If*

$$g(z) = (-z)_\infty \left\{ \sum_{n=0}^{\infty} \frac{z^{2n}q^{n^2-n}}{(-z; q)_n} + \sum_{n=1}^{\infty} z^{-n} q^{1/2\, n(n+1)}(-z^{-1}q)_n \right\},$$

then

$$1/2\ \{g(z)+g(-z)\} = \frac{(q^2;q^2)_\infty(-z^2;q^2)_\infty(-z^{-2}q^2;q^2)_\infty}{(q;q^2)_\infty(q^4;q^{20})_\infty(q^{16};q^{20})_\infty}.$$

PROOF. Here we set

$b_m = q^{m^2}/(q)_{2m},\ a_m = q^{m^2-m}z^{2m},\ c_m = 1/2\ z^{2m}q^{m^2-m}\{(-zq^m;q)_\infty + (zq^m;q)_\infty\}.$

Then

$$\begin{aligned}\sum_{m=0}^{\infty} a_{m+n}b_m &= z^{2n}q^{n^2-n}\sum_{m=0}^{\infty}\frac{q^{2m^2+2mn-m}z^{2m}}{(q)_{2m}} \\ &= 1/2\ z^{2n}q^{n^2-n}\sum_{m=0}^{\infty}\frac{q^{1/2\ m(m-1)+mn}z^m}{(q)_m}(1+(-1)^m) \\ &= c_n,\end{aligned}$$

where the last line follows again from an identity of Euler [**9**, p. 93, Equation (3.22.18)]. It is now easily verified that $1/2\{g(z)+g(-z)\} = \sum_{n=-\infty}^{\infty} c_n$. Finally

$$\sum_{m=0}^{\infty} b_m = \{(q;q^2)_\infty(q^4;q^{20})_\infty(q^{16};q^{20})_\infty\}^{-1},$$

by [**8**, p. 160, Equation (79)], and

$$\sum_{n=-\infty}^{\infty} a_n = (q^2;q^2)_\infty(-z^2;q^2)_\infty(-z^{-2}q^2;q^2)_\infty,$$

by [**9**, p. 86, Equation (3.1.12)].

Just as with Theorem 2, Theorem 3 may also be related to Ramanujan's mock theta functions, and a result similar to Theorem 3 may be obtained by setting $b_m = q^{m^2+2m}/(q)_{2m+1}$, $a_m = q^{m^2-m}z^{2m}$ and utilizing [**8**, p. 162, Equation (96)].

Our final result may be considered a reduction theorem for the general ${}_2\psi_2$ in that we obtain an expansion of the general ${}_2\psi_2$ in a series in which ${}_2\psi_2$'s with one vanishing parameter appear.

THEOREM 4.

$$\begin{aligned}{}_2\psi_2\left[\begin{matrix}a_1, a_3; t, q\\ b_1, b_2\end{matrix}\right] &= \Pi\left[\begin{matrix}b_1b_2/a_1a_3t, b_2/a_3; q\\ b_2;\end{matrix}\right] \\ &\quad\cdot\sum_{m=0}^{\infty}\frac{(b_1/a_1)_m(a_3)_m(b_2/a_3)^m}{(q)_m(b_1)_m}\ {}_2\psi_2\left[\begin{matrix}a_1, a_3q^m; t, q\\ b_1q^m, 0\end{matrix}\right].\end{aligned}$$

PROOF. We let

$$a_m = \frac{(a_1)_m(a_3)_m t^m}{(b_1)_m(b_2)_m}, \qquad b_m = \frac{(-1)^m q^{1/2\, m(m-1)}(b_1b_2/a_1a_3t)^m}{(q)_m},$$

$$c_n = \frac{t^n(a_1)_n(b_2/a_3)_\infty}{(b_2)_\infty} \sum_{m=0}^{\infty} \frac{(b_1/a_1)_m(a_3)_{m+n}(b_2/a_3)^m}{(q)_m(b_1)_{m+n}}.$$

Then $\sum_{m=0}^{\infty} a_{m+n}b_m = c_n$ follows directly from the identity

$$\lim_{a_2\to\infty} {}_3\phi_2\begin{bmatrix} a_1, a_2, a_3; b_1b_2/a_1a_2a_3, q \\ b_1, b_2 \end{bmatrix}$$
$$= \lim_{a_2\to\infty} \prod\begin{bmatrix} b_2/a_3, b_1b_2/a_1a_2; q \\ b_2, b_1b_2/a_1a_2a_3 \end{bmatrix} {}_3\phi_2\begin{bmatrix} b_1/a_1, b_1/a_2, a_3; b_2/a_3, q \\ b_1, b_1b_2/a_1a_2 \end{bmatrix}$$

due to Sears [**7**, p. 174, Equation (10.1)].

Theorem 4 now follows directly from the TL.

It should be obvious from the preceding that many bilateral theorems may be obtained utilizing the TL. We have tried here to exhibit a few of the most interesting applications.

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