A NOTE ON THE BRAUER-SPEISER THEOREM[1]

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ABSTRACT. The Brauer-Speiser theorem asserts that the Schur index of a real-valued complex irreducible character of a finite group is either 1 or 2. In this paper we present a brief proof of this result. From this it follows that the K-central nontrivial division algebra components of group algebras over a real algebraic number field K are quaternions.

THEOREM (BRAUER-SPEISER [2], [5]). *Let χ be a real-valued irreducible complex character of the finite group G. If $\chi(1)$ is odd, then the Schur index $m_Q(\chi)$ of χ over the rationals Q equals* 1. *If $\chi(1)$ is even, then $m_Q(\chi)$ equals* 1 *or* 2.

PROOF. Let H be the direct product of G with itself and let $G_0 \cong G$ be the diagonal of H. We identify G with G_0. Let S be an irreducible complex representation of G affording χ and let $S^{(2)}$ denote the representation of H obtained by taking the outer tensor product of S with itself. $S^{(2)}$ is an irreducible representation of H [4, p. 189]. Denote the character of $S^{(2)}$ by $\chi^{(2)}$ so that $\chi^{(2)}(g_1, g_2) = \chi(g_1)\chi(g_2)$. Thus $\chi^{(2)}|_G = \chi^2$. Let ρ be the 1-character of G. By Frobenius reciprocity and the orthogonality relations,

$$(\rho^H, \chi^{(2)})_H = (\rho, \chi^{(2)}|_G)_G = (\rho, \chi^2)_G = 1, \qquad \text{since } \chi = \bar{\chi}.$$

Since the representation of H affording ρ^H can be realized in Q, it follows from the theory of the Schur index that $m_Q(\chi^{(2)}) = 1$ [4, §70]. Let $K = Q(\chi)$ and let A be the simple component of the group algebra $K(G)$ such that S is an absolutely irreducible constituent of the representation of G afforded by a minimal left ideal of A. Since $K(H) \cong K(G) \otimes_K K(G)$, we see that $S^{(2)}$ is an absolutely irreducible constituent of the representation of H afforded by a minimal left ideal of $A^{(2)} = A \otimes_K A$. Since $m_Q(\chi^{(2)}) = 1$, it follows that $A^{(2)}$ is a complete matrix algebra over K [4, §70]. Thus A has exponent equal to 1 or 2 in the Brauer group of K. Since K is an algebraic number field, the exponent of A equals the index $m_Q(\chi)$ of A. (See [1].) The Brauer-Speiser theorem now follows from the fact that $m_Q(\chi)$ divides $\chi(1)$ [4, Theorem 70.12].

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If we drop the assumption that χ be real valued, the above proof yields the fact that $m_Q(\chi)$ divides $r(\chi^r, \psi)_G$ where ψ is any character afforded by a $Q(\chi)$-representation of G. More general results than this may be found in [3].

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