TWO THEOREMS ON RIEMANN SURFACES WITH NONCYCLIC AUTOMORPHISM GROUPS[1]

ROBERT D. M. ACCOLA

ABSTRACT. Let W be a Riemann surface with finite automorphism group, G. Two formulas are proved which relate the genera of W, W/G and W/H where H ranges over certain subgroups of G. The two theorems are in a sense dual to each other.

Recently the author discovered a theorem concerning Riemann surfaces with automorphism groups admitting partitions.[2] Since the requirement that an automorphism group admit a partition is a special condition, it is not surprising that the result is a special case of a more general theorem. In this note the more general theorem is proven. More interesting perhaps is that this general theorem has in a sense a dual statement. When the automorphism group is noncyclic abelian, the duality is most apparent and seems to be of most significance. The dual theorem is of further interest since it admits obvious generalizations while for the first theorem the generalizations do not seem immediate.

THEOREM 1. *Let W be a Riemann surface of genus p admitting a finite automorphism group G_0. Let $G_1, G_2, \cdots, G_s$ be subgroups of G_0 so that $G_0=\bigcup_{k=1}^s G_k$. For indices $1\leqq i<j<\cdots<k\leqq s$ let $G_{ij\cdots k}=G_i\cap G_j\cap\cdots\cap G_k$. Let p_0, $p_{ij\cdots k}$ be the genera of the quotient Riemann surfaces W/G_0, $W/G_{ij\cdots k}$, and let n_0, $n_{ij\cdots k}$ be the orders of G_0, $G_{ij\cdots k}$. Then*

$$
\begin{aligned}
n_0p_0 = {} & \sum_{1\leq i\leq s} n_ip_i - \sum_{1\leq i<j\leq s} n_{ij}p_{ij} \\
& + \sum_{1\leq i<j<k\leq s} n_{ijk}p_{ijk} - \cdots - (-1)^s n_{12\cdots s}p_{12\cdots s}.
\end{aligned}
\tag{1}
$$

PROOF. For the coverings $W\to W/G_0$ and $W\to W/G_{ij\cdots k}$ the Riemann-Hurwitz formula gives

Received by the editors December 23, 1969.

AMS Subject Classifications. Primary 3045; Secondary 1240.

Key Words and Phrases. Riemann surface, automorphism group, irreducible representation.

[1] Research sponsored by the National Science Foundation GP-7651 and Institute for Advanced Study Grant-In-Aid.

[2] *Riemann surfaces with automorphism groups admitting partitions*, Proc. Amer. Math. Soc. 21 (1969), 477–482.

$$2p - 2 = n_0(2p_0 - 2) + r_0 \tag{2}$$

and

$$2p - 2 = n_{ij\cdots k}(2p_{ij\cdots k} - 2) + r_{ij\cdots k} \tag{3}$$

where r_0, $r_{ij\cdots k}$ are the ramifications of the coverings under consideration. Let $x \in W$ be a branch point of the cover $W \to W/G_0$. The stabilizer of x is a cyclic subgroup $\langle \phi_x \rangle$ of G_0. Let m_{x0} be the order of ϕ_x, and let $m_{xij\cdots k}$ be the order of $\langle \phi_x \rangle \cap G_{ij\cdots k}$. Then the usual counting of elements gives

$$m_{x0} = \sum_i m_{xi} - \sum_{i<j} m_{xij} + \sum_{i<j<k} m_{xijk} - \cdots \tag{4}$$

since $G_0 = \bigcup_{i=1}^{s} G_i$ and so $\langle \phi_x \rangle = \bigcup_i (\langle \phi_x \rangle \cap G_i)$. (This fact is simply a counting of elements in a set and has nothing to do with group theory.) Since

$$1 = \sum_i 1 - \sum_{i<j} 1 + \sum_{i<j<k} 1 - \cdots \tag{5}$$

we have

$$m_{x0} - 1 = \sum_i (m_{xi} - 1) - \sum_{i<j} (m_{xij} - 1) + \sum_{i<j<k} (m_{xijk} - 1) \cdots . \tag{6}$$

But the contribution of x to the ramification of $W \to W/G_{ij\cdots k}$ is $m_{xij\cdots k} - 1$. Summing (6) over all branch points of $W \to W/G_0$ yields

$$r_0 = \sum_i r_i - \sum_{i<j} r_{ij} + \sum_{i<j<k} r_{ijk} - \cdots . \tag{7}$$

Now substitute equations (2) and (3) into (7).

$$\begin{aligned}(2p - 2) - n_0(2p_0 - 2) = &\sum_i [(2p - 2) - n_i(2p_i - 2)] \\ &- \sum_{i<j} [(2p - 2) - n_{ij}(2p_{ij} - 2)] + \cdots .\end{aligned} \tag{8}$$

Eliminate the terms $2p-2$ from each summand by (5). Divide by two. Note that

$$n_0 = \sum n_i - \sum n_{ij} + \sum n_{ijk} - \cdots \tag{9}$$

and so eliminate these terms. (8) then reduces to (1).

COROLLARY. *Suppose for $0 \neq i \neq j \neq 0$ it happens that $G_i \cap G_j = \langle e \rangle$. G_0 is then said to admit a partition. In that case $p_{ij\cdots k} = p$ and $n_{ij\cdots k} = 1$ for indices with two or more numbers. Formula* (1) *becomes*

$$(10) \qquad n_0 p_0 = \sum_{i=1}^{s} n_i p_i + p\left[-\binom{s}{2} + \binom{s}{3} - \binom{s}{4} + \cdots\right]$$

or

$$(11) \qquad n_0 p_0 = \sum n_i p_i + (1 - s)p.$$

THEOREM 2. *Let W be a Riemann surface of genus p admitting a finite automorphism group G_0. Let $G_1, G_2, \cdots, G_s$ be normal subgroups. Suppose for each irreducible complex representation of G_0, χ, there is an i so that $G_i \subset$ kernel χ. For indices $1 \leqq i < j < \cdots < k \leqq s$ let $G_{ij\cdots k} = G_i G_j \cdots G_k$ and let $p_{ij\cdots k}$ be the genus of $W/G_{ij\cdots k}$. Then*

$$(12) \quad p = \sum_{1 \leq i \leq s} p_i - \sum_{1 \leq i < j \leq s} p_{ij} + \sum_{1 \leq i < j < k \leq s} p_{ijk} - \cdots - (-1)^s p_{12\cdots s}.$$

PROOF. Let A denote the space of analytic differentials on W. G_0 acting on A gives a complex representation μ. Let R be the set of complex irreducible representations of G_0. Since μ is completely reducible, A is the direct sum of subspaces

$$(13) \qquad A = \sum_{\chi \in R} \sum_{j=1}^{t_\chi} A_{\chi j}$$

where G_0 acts irreducibly on $A_{\chi j}$ for each j and gives the representation χ. χ occurs in μ t_χ times. If $q_\chi = t_\chi(\dim A_{\chi j})$, then (13) yields

$$(14) \qquad p = \sum_{\chi \in R} q_\chi.$$

If N is a normal subgroup of G_0 let $\hat{N} = \{\chi \in R \mid T \in N \Rightarrow \chi(T) = 1\}$. That is $\hat{N}$ is the set of irreducible representations which include N in their kernels. The proof of (12) depends on the following lemma.

LEMMA.

$$(15) \qquad p_{ij\cdots k} = \sum_{\chi \in \hat{G}_{ij\cdots k}} q_\chi.$$

Assuming the lemma, let us prove (12). Fix $\chi \in R$. Suppose $G_{i_1}, G_{i_2}, \cdots, G_{i_l}$ are included in ker χ. Then $G_{ij\cdots k} \subset$ ker χ if and only if the set of indices $\{i, j, \cdots, k\}$ is included in $\{i_1, i_2, \cdots, i_l\}$. Now apply the lemma to the right-hand side of (12). q_χ is counted in $\sum p_i$ l times; in $\sum p_{ij}$ $\binom{l}{2}$ times; in $\sum p_{ijk}$ $\binom{l}{3}$ times; etc. Thus q_χ is counted $\binom{l}{1} - \binom{l}{2} + \binom{l}{3} - \cdots$ times, that is, once. Since each $\chi \in R$ occurs at least once, the right-hand side of (12) equals $\sum_{\chi \in R} q_\chi$. By (14) the formula is proven.

PROOF OF THE LEMMA. We now prove: if N is normal in G_0, then the genus of W/N, call it p_N, satisfies

$$p_N = \sum_{\chi \in \hat{N}} q_\chi.$$

Let A_N be the space of analytic differentials in A which are lifts of differentials from W/N. Then $p_N = \dim A_N$. If $T \in G_0$, denote by T^* the natural action of T on A. Then A_N is the subspace of A which N leaves pointwise fixed. The lemma follows from the assertion

$$A_N = \sum_{\chi \in \hat{N}} \sum_{j=1}^{t_\chi} A_{\chi j}. \tag{16}$$

We prove the double inclusion in (16) in two steps.

(i) $\sum_{\chi,j} A_{\chi j} \subset A_N$. Suppose $\chi \in \hat{N}$ and $T \in N$. Then T^* acting on $A_{\chi j}$ is trivial. Thus if $\sigma \in A_{\chi j}$, $T^*\sigma = \sigma$; that is, $\sigma \in A_N$.

(ii) $A_N \subset \sum_{\chi,j} A_{\chi j}$. Fix $\sigma \in A_N$. By (13)

$$\sigma = \sum_{\chi \in R} \sum_{j=1}^{t_\chi} \sigma_{\chi j} \tag{17}$$

where $\sigma_{\chi j} \in A_{\chi j}$. For $T \in N$ we have

$$\sum \sigma_{\chi j} = \sigma = T^*\sigma = \sum T^*\sigma_{\chi j}.$$

Since (13) gives a direct sum decomposition of A we have $T^*\sigma_{\chi j} = \sigma_{\chi j}$ for all χ, j. If $\sigma_{\chi j} \neq 0$, then $A_{\chi j}$ is spanned by

$$\{S^*\sigma_{\chi j} \mid S \in G_0\}$$

since G_0 acts irreducibly on $A_{\chi j}$. But if $T \in N$, $T^*S^*\sigma_{\chi j} = S^*(S^{-1}TS)^*\sigma_{\chi j} = S^*\sigma_{\chi j}$. Thus if $\sigma_{\chi j} \neq 0$, T acts trivially on $A_{\chi j}$. Thus $\chi \in \hat{N}$. q.e.d.

APPLICATION. Let $G_0 = Z_m \times Z_m \times \cdots \times Z_m$ k times where m is a prime integer. The irreducible representations (characters) of G_0 correspond to the $s = (m^k - 1)/(m-1)$ (normal) subgroups of index m. Let these be the subgroups, $G_1, G_2, \cdots, G_s$ of Theorem 2. Then $G_iG_j = G_0$ for $i \neq j$ and so $p_{ij\cdots k} = p_0$ for any index with two or more numbers. p_0 is the genus of W/G_0. (12) becomes

$$p = \sum_{i=1}^{s} p_i + p_0\left[-\binom{s}{2} + \binom{s}{3} - \binom{s}{4} + \cdots\right]$$

or $p + (s-1)p_0 = \sum p_i$. If $p_0 = 0$, then $p = \sum_{i=1}^{s} p_i$.

REMARKS. If $p_i = p$ in (12) for some i, it is easily seen that formula

(12) says nothing. Thus the theorem seems of little interest if G_0 admits a faithful irreducible representation.

It should be noted that the method of Theorem 2 can be applied to any complex vector space associated with W on which G_0 acts. Examples of interest are quadratic differentials and meromorphic functions (or differentials) which are multiples of a fixed divisor invariant under G_0.

To illustrate this last point, suppose W admits an automorphism group G_0 with subgroups $G_1, \cdots, G_s$ satisfying the hypotheses of Theorem 2. Let D_0 be a divisor of degree d on W_0. Let D, $D_{ij\cdots k}$ be divisors of degree $n_0 d$, $n_0 d/n_{ij\cdots k}$ on W, $W/G_{ij\cdots k}$ obtained by lifting D_0 to the surfaces involved and where necessary counting the branch points according to multiplicity. Let l, $l_{ij\cdots k}$ be the dimensions of the spaces of meromorphic functions on W, $W/G_{ij\cdots k}$ which are multiples of D, $D_{ij\cdots k}$. Then

$$l = \sum l_i - \sum l_{ij} + \sum l_{ijk} - \cdots - (-1)^s l_{12\cdots s}.$$

Brown University, Providence, Rhode Island 02912