LIAPUNOV FUNCTIONS AND GLOBAL EXISTENCE WITHOUT UNIQUENESS

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ABSTRACT. In a recent paper J. Kato and A. Strauss characterized the global existence of solutions of an ordinary differential equation in terms of Liapunov functions in which they assumed the right hand side of the differential equation is locally Lipschitz. In the present paper a characterization of global existence of an ordinary differential equation is found in which the right hand side is merely continuous. The construction of the Liapunov functions depend heavily upon the properties of solution funnels due to the nonuniqueness of solutions.

1. **Introduction.** In most of the theory dealing with the construction of Liapunov functions for the ordinary differential equation

(E) $$\dot{x} = f(t, x),$$

one assumes that f is locally Lipschitz in order to obtain a locally Lipschitz Liapunov function. In this paper we consider (E) under the assumption that $f: R \times R^n \to R^n$ is merely continuous and provide necessary and sufficient conditions for the global existence of solutions of (E) in terms of Liapunov functions which depend upon solution funnels. Kato and Strauss [1] have considered this problem assuming f is locally Lipschitz. Their results, as we shall show in an example, do not hold when f is continuous even if we do not require the Liapunov function to be continuous.

2. **Preliminaries.** Let R^n denote Euclidean n-space. $||$ will denote the Euclidean norm. For $x, y \in R^n$ define $d(x, y) = |x - y|$. For a set $S \subset R^n$, $S \neq \varnothing$ define

$$d(x, S) = \inf\{d(x, y) \mid y \in S\}.$$

Let T be another subset of R^n, $T \neq \varnothing$. Then we define

$$d(T, S) = \inf\{d(x, S) \mid x \in T\}.$$

Denote a solution of (E) through the point (t_0, x_0) by $x(\cdot, t_0, x_0)$. A solution through (t_0, x_0) exists in the future if $x(t, t_0, x_0)$ exists for all

$t > t_0$ and exists in the past if $x(t, t_0, x_0)$ exists for all $t < t_0$. A solution exists forever if it exists in the past and in the future.

For $(t_0, x_0) \in R \times R^n$ define the positive and negative solution funnels as

$$F^+_{t_0, x_0} = \{(t, x(t)) : t \geqq t_0, x(t_0) = x_0\} \subset R^{n+1}$$

and

$$F^-_{t_0, x_0} = \{(t, x(t)) : t \leqq t_0, x(t_0) = x_0\} \subset R^{n+1}$$

respectively, where $\dot{x}(t) = f(t, x(t))$. The solution funnel through (t_0, x_0), denoted by F_{t_0, x_0}, is defined as

$$F_{t_0, x_0} = F^+_{t_0, x_0} \cup F^-_{t_0, x_0}.$$

We shall use the notation $F_{t_0, x_0}[a, b]$ to denote the restriction of the solution funnel to the t-interval $[a, b]$. The τ-cross-section, denoted by $F_{t_0, x_0}(\tau)$, is the subset of R^n formed by the intersection of F_{t_0, x_0} and the hyperplane $t = \tau$; that is,

$$F_{t_0, x_0}(\tau) = F_{t_0, x_0} \cap (\{\tau\} \times R^n).$$

If A is any set in R^n then we can define the solution funnel through $t_0 \times A$ restricted to any interval $[a, b]$ as

$$F_{t_0, A}[a, b] = \bigcup_{p \in A} F_{t_0, p}[a, b]$$

and the τ-cross-section of the funnel by

$$F_{t_0, A}(\tau) = \bigcup_{p \in A} F_{t_0, p}(\tau).$$

We shall need the following known results dealing with solution funnels.

LEMMA 1 [2]. *If A is a compact set and if all solutions through $t_0 \times A$ exist on the interval $[t_0, \tau]$ then $F_{t_0, A}[t_0, \tau]$ and $F_{t_0, A}(\tau)$ are both compact.*

Let ρ denote the Hausdorff metric on the class $\mathcal{K}$ of all nonempty compact sets of R^n.

LEMMA 2 [2]. *Consider the funnel of solutions of* (E) *through the point $(t_0, x_0) = p$. Suppose all solutions through p exist forever. Then the mapping F_p, where $F_p(t)$ is the t-cross-section of the solution funnel through p, has the property that $F_p : R \to \mathcal{K}$ is continuous in the Hausdorff metric topology.*

LEMMA 3 [3]. *Let f be a continuous mapping on an open set $D \subset R$* $\times R^n$. *Let $(\tau_0, \xi_0) \in D$ and suppose all solutions of* (E) *through (τ_0, ξ_0) exist on $[a, b]$, $\tau_0 \in [a, b]$. Then for each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, such that if $d((\tau, \xi), (\tau_0, \xi_0)) < \delta$, then for each solution $x(\cdot, \tau, \xi)$ of* (E), *there exists a solution $x(\cdot, \tau_0, \xi_0)$ of* (E) *through (τ_0, ξ_0) such that $|x(t, \tau, \xi) - x(t, \tau_0, \xi_0)| < \epsilon$ for all $t \in [a, b]$.*

3. **Results.** As mentioned before Kato and Strauss [1] provided necessary and sufficient conditions for the existence of solutions of (E) on $[t_0, \infty)$ in terms of Liapunov functions assuming f is locally Lipschitz. In the next theorem we only assume f is continuous and construct a Liapunov function similar to Yoshizawa's [4].

THEOREM 1. *All solutions of* (E) *exist in the future if and only if there exists a function $V: R \times R^n \rightarrow R$ such that*
   (a) $V(t, x) \rightarrow \infty$ *as $|x| \rightarrow \infty$ uniformly for t in compact sets, and*
   (b) $V(t, x(t))$ *is a nonincreasing function of t for all solutions $x(t)$.*

Proof. Suppose there exists a function $V(t, x)$ satisfying (a) and (b) and not all solutions of (E) exist in the future. Then there exists a point (t_0, x_0) and a solution $x(\cdot, t_0, x_0)$ such that $|x(t, t_0, x_0)| \rightarrow \infty$ as $t \rightarrow T < \infty$. Consider the interval $[t_0, T]$. By (a),

$$V(t, x(t, t_0, x_0)) \rightarrow \infty \qquad \text{as } t \rightarrow T,$$

and by (b) $V(t_0, x_0) \geq V(t, x(t, t_0, x_0))$. Hence, $V(t_0, x_0)$ is unbounded, a contradiction.

Conversely, we define

$$V(t, x) = \inf_{\tau}(|x(\tau, t, x)|) \quad \text{for all} \quad x(\cdot) \in F^-_{t,x}; \qquad \tau \in [0, t] \cap I^*,$$

$$= \sup_{\tau}(|x(\tau, t, x)|) \quad \text{for all} \quad x(\cdot) \in F^+_{t,x}; \qquad \tau \in [t, 0],$$

where I^* is the largest interval to the left of t on which $x(\tau, t, x)$ is defined for $t > 0$.

For any solution $x(t)$, consider the expression $V(t+h, x(t+h)) - V(t, x)$ where $h > 0$. By the definition of V we have $V(t+h, x(t+h)) \leq V(t, x)$; that is $V(t, x(t))$ is a nonincreasing function in t, thus satisfying (b).

Assume V does not satisfy (a). Then there exist M and $T > 0$ and sequences $\{t_n\}$, $\{x_n\}$ such that $|V(t_n, x_n)| \leq M$ as $|x_n| \rightarrow \infty$ and $-T \leq t_n \leq T$. Since $t \leq 0$ implies $V(t, x) \geq |x|$ we must have $t_n > 0$ for sufficiently large n. By the definition of V, there exists a sequence of

points. $\{\tau_n\}$ such that $\tau_n \leqq t_n$ and a sequence of solutions $\{y_n(\cdot, t_n, x_n)\}$ such that

$$\left| y_n(\tau_n, t_n, x_n) \right| \leqq 2M \quad \text{for } 0 \leqq \tau_n \leqq t_n.$$

Since $\{t_n\}$, $\{\tau_n\}$, and $\{\left| y_n(\tau_n, t_n, x_n) \right|\}$ lie in compact sets, we can assume without loss of generality that

$$t_n \to t_0, \quad \tau_n \to \tau_0, \quad \text{and} \quad y_n(\tau_n, t_n, x_n) \to y_0 \qquad \text{as } n \to \infty.$$

We consider the solution funnel F_{τ_0, y_0} and denote F_{τ_0, y_0} by F. There exists $\delta > 0$ such that all solutions exist on $[\tau_0 - \delta, \infty)$, and for large n we have $t_n \geqq \tau_n > \tau_0 - \delta$. By an application of Lemma 3 we have that for each $\epsilon > 0$ there exists $N(\epsilon)$ such that

$$d(y_n(s, \tau_n, y_n(\tau_n, t_n, x_n)), F(s)) < \epsilon$$

for $s \in [\tau_0 - \delta, 2T]$ and $n \geqq N(\epsilon)$. In particular if we set $s = t_n$ we have

$$(3.1) \qquad\qquad d(x_n, F(t_n)) < \epsilon, \qquad n \geqq N(\epsilon).$$

Using the properties of the Hausdorff metric we obtain

$$(3.2) \qquad\qquad d(x_n, F(t_0)) \leqq d(x_n, F(t_n)) + \rho(F(t_n), F(t_0)).$$

Using Lemma 2, we have for n sufficiently large that $\rho(F(t_n), F(t_0)) < \epsilon$. Then using (3.1) and (3.2) we obtain for n sufficiently large

$$(3.3) \qquad\qquad d(x_n, F(t_0)) \leqq 2\epsilon.$$

However, by Lemma 1, $F(t_0)$ is compact; and with (3.3), we have that $\{x_n\}$ is bounded, a contradiction.

COROLLARY 1. *All solutions of* (E) *exist in the past if and only if there exists a function* $V: R \times R^n \to R$ *such that V satisfies condition* (a) *and*

(b¹) $\qquad\qquad V(t, x(t)) \qquad$ *is a nondecreasing function of* t.

PROOF. The sufficiency follows from a contradiction similar to that in Theorem 1.

For the necessity, we define

$$V(t, x) = \sup_{0 \leqq \tau \leqq t} \left(\left| x(\tau, t, x) \right| \right) \quad \text{for all} \quad x(\cdot) \in F^-_{t,x}, \qquad \text{if } t > 0,$$

$$= \inf_{t \leqq \tau \leqq 0} \left(\left| x(\tau, t, x) \right| \right) \quad \text{for all} \quad x(\cdot) \in F^+_{t,x}, \qquad \text{if } t \leqq 0.$$

Once again using techniques similar to those in Theorem 1 we can verify that V satisfies (a) and (b¹).

In view of Theorem 1 and Corollary 1, a reasonable conjecture concerning the global existence of solutions of (E) would be that all solutions of (E) exist forever if and only if there exists a function V satisfying (a) and in addition V is constant along all solutions. This conjecture is true when solutions are unique (Kato and Strauss [1]). The following example, however, shows that this may not be true when solutions are not unique.

Consider the scalar equation

(S)
$$\dot{x} = (x - n)^{1/2}(n + 1 - x)^{1/2} \qquad n \le x \le n + 1$$
$$= 0 \qquad\qquad\qquad\qquad\qquad x < 0,$$

where $n = 0, 1, \cdots$. All solutions of (S) exist forever. We shall suppose that there exists a function V which is constant along solutions and show that V cannot satisfy condition (a). Consider the point $t = 0$, $x = 1$; we shall show $V(0, 1) = V(0, n)$ for all n. Through the point $(0, 1)$ there exists a solution $x(\cdot, 0, 1)$ and a point $\tau(n)$ such that $x(\tau(n), 0, 1) = n$. Hence $V(\tau(n), n) = V(0, 1)$. Since $x \equiv n$ is a solution we have $V(0, n) = V(0, 1)$. Hence V does not satisfy (a).

By imposing a growth condition on V along solutions we arrive at the following theorem concerning the global existence of solutions.

THEOREM 2. *All solutions of* (E) *exist forever if and only if there exists a function* $V : R \times R^n \to R$ *such that V satisfies* (a) *and in addition for every point* $p = (t_0, x_0) \in R \times R^n$ *and for all solutions* $x(\cdot) \in F_p$ *we have*

(c)
$$V(t, x(t, t_0, x_0)) \le r_p(t)$$

where $r_p : R \to R$, *satisfies* $r_p(t_0) = V(t_0, x_0)$, *and is bounded above for t in compact sets of R.*

PROOF. Suppose there exists a function $V(t, x)$ satisfying (a) and (c). Assume there exists a point $q = (t_1, x_1)$ and a solution $x(\cdot, t_1, x_1)$ such that $x(\cdot, t_1, x_1)$ does not exist forever. Then there either exists an α such that $t_1 < \alpha < \infty$ and

(3.4)
$$|x(t, t_1, x_1)| \to \infty \qquad \text{as } t \to \alpha^-,$$

or there exists a β such that $-\infty < \beta < t_1$ and

(3.5)
$$|x(t, t_1, x_1)| \to \infty \qquad \text{as } t \to \beta^+.$$

Using (3.4) and (a) we have for $t \in [t_1, \alpha]$

(3.6)
$$V(t, x(t, t_1, x_1)) \to \infty \qquad \text{as } t \to \alpha^-.$$

By (c), $r_q(t) \ge V(t, x(t, t_1, x_1))$ and $r_q(t)$ is bounded on $[t_1, \alpha]$, a con-

tradiction to (3.6). A similar contradiction holds if (3.5) is satisfied. Conversely, we define

$$V(t, x) = \sup_{0 \leq \tau \leq t} (\,|\,x(\tau, t, x)\,|\,) \quad \text{for all} \quad x(\cdot) \in F_{t,x}^{-}, \quad \text{if } t > 0,$$

$$= \sup_{t \leq \tau \leq 0} (\,|\,x(\tau, t, x)\,|\,) \quad \text{for all} \quad x(\cdot) \in F_{t,x}^{+}, \quad \text{if } t \leq 0.$$

Since all solutions exist forever, we have as a consequence of Lemma 1 that V is well defined. Morever $V(t, x) \geq |x|$; hence, V satisfies (a).

Given any point $p = (t_0, x_0)$ we now construct $r_p(t)$. First let $t_0 \geq 0$. Then for $0 \leq t \leq t_0$, define $r_p(t) = V(t_0, x_0)$. For $t < 0$, we consider the set $A = F_p(t)$ and the set $F_{t,A}[t, 0]$. Since $F_{t,A}[t, 0]$ is compact from Lemma 1, we have that the $\sup_{t \leq \tau \leq 0}(|\,F_{t,A}(\tau)\,|)$ is finite. Define

$$r_p(t) = \sup_{t \leq \tau \leq 0} (\,|\,F_{t,A}(\tau)\,|\,) \quad \text{for } t < 0.$$

Finally for $t \geq t_0$ define

$$r_p(t) = \sup_{0 \leq \tau \leq t} (\,|\,F_{t,A}(\tau)\,|\,),$$

where again $A = F_p(t)$.

Now let $t_0 < 0$. In a similar manner we define

$$r_p(t) = V(t_0, x_0) \quad \text{for} \quad t_0 \leq t \leq 0,$$

$$= \sup_{0 \leq \tau \leq t} (\,|\,F_{t,A}(\tau)\,|\,), \quad \text{for} \quad t > 0,$$

$$= \sup_{t \leq \tau \leq 0} (\,|\,F_{t,A}(\tau)\,|\,), \quad \text{for} \quad t < t_0,$$

where once again $A = F_p(t)$.

By the definitions of $r_p(t)$ and $V(t, x)$ we have that for any point $p = (t_0, x_0)$, $V(t, x(t, t_0, x_0)) \leq r_p(t)$ for all $t \in R$ and all $x(\cdot) \in F_{t_0, x_0}$. We also notice that for $t_0 \geq 0$, $r_p(t)$ is a nondecreasing function on $[t_0, \infty)$, constant on $[0, t_0]$, and nonincreasing on $(-\infty, 0]$. Similarly, for $t_0 \leq 0$, we have $r_p(t)$ is nondecreasing on $[0, \infty)$, constant on $[t_0, 0]$ and nonincreasing on $(-\infty, t_0]$. For any compact set $K \in R$ there exists a $T > 0$ such that $K \in [-T, T]$. For $t \in [-T, T]$ we have by the above comments that

$$r_p(t) \leq \max(r_p(T), r_p(-T)).$$

Hence $r_p(t)$ is bounded on K and therefore we have that condition (c) is satisfied which completes the proof.

REMARK. We can in fact show that $r_p(t)$ is continuous by applying Lemmas 2 and 3.

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