SUBALGEBRAS OF $L_\infty$ OF THE CIRCLE GROUP

LEON BROWN

Abstract. In this note, we prove a theorem about subalgebras of a Banach algebra. Thus a theorem of J. P. Kahane and Y. Katznelson implies a theorem of R. Salem and these theorems imply that a number of subspaces of $L_\infty$ of the circle group are not algebras.

An elementary application of the closed graph theorem and the uniform boundedness principle yields the following theorem:

**Theorem 1.** Let $(E, || ||)$ be a normed algebra, $A$ a subspace (not necessarily closed) of $E$. Let $|| ||_1$ be a norm on $A$ such that $||f||_1 \geq K||f||$, and $(A, || ||_1)$ be a Banach space. Then if $A$ is an algebra then multiplication is $|| ||_1$ continuous, i.e., there exists $M$ such that $||fg||_1 \leq M||f||_1||g||_1$. Thus if $A$ is an algebra there exists a $|| ||_2$ equivalent to $|| ||_1$ such that $(A, || ||_2)$ is a Banach Algebra.

Proof. (a) Let $L_f: A \to A$ be defined as follows: $L_fg = fg$. Then $L_f$ is a continuous map. Let $g_n \in A$, $||g_n - g||_1 \to 0$ and $||L_fg_n - h||_1 \to 0$. This implies that $||g_n - g|| \to 0$, $||fg_n - h|| \to 0$ and thus $||fg_n - fg|| \leq ||f|| ||g_n - g|| \to 0$ and $h = fg = L_fg$. The closed graph theorem implies $L_f$ is continuous.

(b) If $||f||_1 \leq 1$, then $||L_fg||_1 = ||fg||_1 = ||L_o f||_1 \leq ||L_o||_1 ||f||_1 \leq ||L_o||_1$. Therefore by the uniform boundedness principle there exists an $M$ such that $||L_o||_1 \leq M$ if $||f||_1 \leq 1$. Consequently $||fg||_1 = ||f|| ||g||_1 \leq M||f||_1||g||_1$ and this completes the proof of the theorem.

We note that with an appropriate modification of the proof one can prove a similar theorem to the one above for some Banach spaces $E$ (for example, $L_p(X, \Sigma, \mu)$ where $(X, \Sigma, \mu)$ is a measure space).

Let $L_\infty = L_\infty(\Gamma)$ be the Banach algebra of bounded measurable functions on $\Gamma$ with norm $||f||_\infty = \sup_{t \in \Gamma} |f(t)|$ and $C = C(\Gamma)$ is the closed subspace of $L_\infty$ of continuous functions on $\Gamma$. We set

$$S(f) = \sum_{n=\infty}^\infty \int f_n e^{int}$$

the Fourier series of $f$;

$$S_N(f, t) = \sum_{n=-N}^N \int f_n e^{int}$$

the "symmetric Fourier sum" of $f$;

Received by the editors December 8, 1969.

*AMS Subject Classifications.* Primary 4211; Secondary 3030.

*Key Words and Phrases.* Subalgebra, uniform convergence, Fourier series.
\[ S_{MN}(f, t) = \sum_{m} f_n e^{int} \] the "nonsymmetric sum" of \( f \).

With the norm \( \|f\|_U = \sup_{N} ||S_N(f, \cdot)\|_\infty \), we consider the following subspaces of \( L_\infty \):
(a) \( U \) is the set of \( f \) such that \( \|f\|_U < +\infty \).
(b) \( U \) is the set of \( f \) whose Fourier series is uniformly convergent.
(c) \( U \cap C > U \).

With the norm \( \|f\|_{U^*} = \sup_{M,N} ||S_{MN}(f, \cdot)\|_\infty \) we consider the following subspaces of \( L_\infty \):
(a) \( U^*_0 \) is the set of \( f \) such that \( \|f\|_{U^*} < +\infty \);
(b) \( U^* \) is the set of \( f \) whose nonsymmetric Fourier series converges uniformly;
(c) \( U^*_0 \) is the closed subspace of \( U^*_0 \) (or \( U_0 \)) which are Taylor series, i.e., \( f_n = 0 \) for \( n < 0 \);
(d) \( U^T = U^*_0 \cap U^* \) (or \( U^*_0 \cap U \));
(e) \( U^*_0 \cap C \);
(f) \( U^*_0 \cap C \);
(g) \( U^T \cap C \).

We observe that all of these spaces with their respective norm are Banach spaces.

In [2], R. Salem proved that \( U \) is not an algebra. In [1], J. P. Kahane and Y. Katznelson proved the stronger result that \( U^* \) and \( U^T \) are not algebras. Using these theorems we are able to prove the following corollaries:

**Corollary 1.** Any closed subspace of \( U_0 \) which contains \( 1, e^{i\theta}, \) and \( e^{-i\theta} \) is not an algebra.

**Proof.** If it is an algebra then by Theorem 1 there exists an \( M \) such that \( \|PQ\|_U \leq M\|P\|_U\|Q\|_U \) for trigonometric polynomials \( P \) and \( Q \). Since trigonometric polynomials are dense in \( U \), a simple argument would show that \( U \) is an algebra. This contradicts Salem's theorem.

**Corollary 2.** \( U_0, U, C \cap U_0 \) are not algebras.

**Corollary 3.** Any closed subspace of \( U^*_0 \) which contains \( 1 \) and \( z = e^{i\theta} \) is not an algebra.

**Proof.** We observe that if \( P \) is a trigonometric polynomial then \( \|e^{i\theta}P\|_{U^*} = ||P\|_{U^*} \). As in the proof of Corollary 1, we show that \( U^* \) is an algebra which is a contradiction.

**Corollary 4.** \( U^*_0, U^*, U^T_0, U^T, U^*_0 \cap C, U^T_0 \cap C, \) and \( U^T \cap C \) are not algebras.
We conclude with the observation that one can prove that if some closed subspace of $U_0$ which contains 1 and $z$ is not an algebra then none of the subspaces are algebras. This observation could possibly lead to a simpler proof of Kahane and Katznelson's theorem.

References
