THE ARENS PRODUCT AND DUALITY IN $B^*$-ALGEBRAS

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Abstract. Let $A$ be a $B^*$-algebra, $A^{**}$ its second conjugate space and $\pi$ the canonical embedding of $A$ into $A^{**}$. $A^{**}$ is a $B^*$-algebra under the Arens product. Our main result states that $A$ is a dual algebra if and only if $\pi(A)$ is a two-sided ideal of $A^{**}$. Gulick has shown that for a commutative $A$, $\pi(A)$ is an ideal if and only if the carrier space of $A$ is discrete. As this is equivalent to $A$ being a dual algebra, Gulick's result thus carries over to the general $B^*$-algebra.

1. Introduction. Let $A$ be a (complex) commutative $B^*$-algebra and let $\Delta$ be the set of all nonzero multiplicative linear functionals in $A^*$, the conjugate space of $A$. Let $A'$ be the closed span of $\Delta$ in $A^*$ and let $A''=A'^*$. Let $\pi'$ be the embedding of $A$ into $A''$ given by $\pi'(x)=\pi(x)|_{A'}$, the restriction of $\pi(x)$ to $A'$. Birtel [2] has introduced a product in $A''$ under which $A''$ is a commutative Banach algebra. It follows that the multiplier algebra $M(A)$ can be isometrically embedded in $A''$. We make use of $A''$, $A^{**}$ and $M(A)$ to obtain several characterizations of duality for $A$ which we gather together in Theorem 4.2.

2. The multiplier algebra. Let $A$ be a semisimple Banach algebra. A mapping $T$ on $A$ into itself is called a multiplier if $(Tx)y=x(Ty)$ for all $x, y \in A$. It is easy to see that $T$ is a bounded linear operator on $A$ and that $M(A)$, the set of all multipliers on $A$, is a closed commutative subalgebra of the Banach algebra $B(A)$ of all bounded linear operators on $A$ into itself under the usual operator bound norm. $M(A)$ is called the multiplier algebra of $A$. It is easily shown that $M(A)$ is complete under its strong operator topology (i.e., the topology on $M(A)$ generated by the seminorms $T \mapsto \|Tx\|$, $x \in A$). From now on we shall call the strong operator topology on $M(A)$ the strict topology on $M(A)$ [12]. All algebras and vector spaces under consideration are over the complex field $C$.

Let $A$ be a semisimple commutative Banach algebra. Then $A$ can be identified as an ideal of $M(A)$. In what follows we shall always consider $A$ as a subalgebra of $M(A)$. $A$ is strictly dense in $M(A)$ if and only if $A$ has an approximate identity (see [12]). Let $\Omega$ be the carrier
space of \( A \) and \( \hat{A} \) the function algebra on \( \Omega \) isomorphic to \( A \) in the Gelfand theory. Then \( M(A) \) can be identified with the set of all complex-valued functions \( f \) on \( \Omega \) such that \( f\hat{A} \subset \hat{A} \). The functions \( f \) are continuous and if \( T \) is the multiplier corresponding to \( f \) then the sup norm \( \|f\|_\infty \leq \|T\| \) [12, Theorem 3.1].

**Lemma 2.1.** Let \( A \) be a semisimple commutative Banach algebra with an approximate identity and let \( I \) be an ideal of \( A \). Then \( I \) is dense in \( A \) if and only if it is strictly dense in \( M(A) \).

**Proof.** Since \( A \) has an approximate identity, \( A \) is isometrically isomorphic to a subalgebra of \( M(A) \). Let \( \text{cl}(I) \) and \( \text{cl}_s(I) \) denote the norm closure and strict closure of \( I \) in \( M(A) \), respectively. Suppose \( \text{cl}(I) = A \). Since the norm topology is finer than the strict topology on \( M(A) \), we have \( A = \text{cl}(I) \subset \text{cl}_s(I) \). Since \( A \) has an approximate identity, \( \text{cl}_s(A) = M(A) \). Hence \( \text{cl}_s(I) = M(A) \). Conversely suppose \( \text{cl}_s(I) = M(A) \). Let \( x \in A \) and let \( \{x_\beta\} \) be a net in \( I \) converging to \( x \) in the strict topology. Then \( \lim_\beta \|x_\beta e_a - xe_a\| = 0 \), for each \( e_a \). Since \( x_\beta e_a \in I \), we have \( xe_a \in \text{cl}(I) \). Therefore \( x \in \text{cl}(I) \) and so \( \text{cl}(I) = A \).

3. The algebras \( A^{**} \) and \( A'' \). Let \( A \) be a \( B^* \)-algebra. It is well known that the two Arens products defined in \( A^{**} \) coincide [4, Theorem 7.1]. For completeness we sketch the construction of one of the Arens products in \( A^{**} \) which we shall use throughout. We do this in stages as follows. (See [1], [4], [6].) Let \( x, y \in A, f \in A^*, F, G \in A^{**} \).

   (i) Define \( f \star x \) by \( (f \star x)y = f(xy) \). \( f \star x \in A^* \).
   (ii) Define \( G \star f \) by \( (G \star f)x = G(f \star x) \). \( G \star f \in A^* \).
   (iii) Define \( F \star G \) by \( (F \star G)f = F(G \star f) \). \( F \star G \in A^{**} \).

\( A^{**} \) is a \( B^* \)-algebra under this product [4, Theorem 7.1] and, when \( A \) is embedded canonically in \( A^{**} \), it agrees with the given product on \( A \) [1].

Now let \( A \) be a commutative \( B^* \)-algebra. Following Birtel [2] we define a product on \( A'' \) given in stages as follows. Let \( x \in A, f_i \in \Delta, F, G \in A'', \alpha_i \in \mathbb{C} \). (All sums are finite.)

1. Let \( (\sum \alpha_i f_i) \circ x = \sum \alpha_i f_i(x)f_i \).
2. Let \( F \circ (\sum \alpha_i f_i) = \sum \alpha_i F(f_i)f_i \).
3. Let \( F \circ G \) be given by \( F \circ G(\sum \alpha_i f_i) = \sum \alpha_i F(f_i)G(f_i) \). \( F \circ G \) is clearly a continuous linear functional on the span of \( \Delta \) and therefore can be uniquely extended to a linear functional on all of \( A' \). We denote this extension by the same symbol \( F \circ G \). The multiplication thus defined on \( A'' \) is commutative and \( \|F \circ G\| \leq \|F\| \|G\| \) and \( \pi' \) is an isomorphism of \( A \) into \( A'' \), taking the product \( xy \) into \( \pi'(x) \circ \pi'(y) \).
4. Duality in a commutative $B^*$-algebra. Let $A$ be a commutative $B^*$-algebra with carrier space $\Omega$. Since $A$ has a bounded approximate identity and is a supremum norm algebra, $M(A)$ can be isometrically embedded in $A''$ [2], [12]. We shall assume in what follows that $M(A) \subseteq A''$.

**Lemma 4.1.** Let $\phi$ be an element of $\Omega$ and let $\Phi$ be a subset of $\Omega$ such that $\phi \notin \Phi$. Let $M$ be the closed subspace of $A'$ spanned by the elements of $\Phi$. Then $\phi \notin M$.

**Proof.** Suppose $\phi \in M$. Then there exists $\phi_i \in \Phi$ and $\alpha_i \in C$ ($i = 1, 2, \ldots, n$) such that

$$\left\| \phi - \sum_{i=1}^{n} \alpha_i \phi_i \right\| < 1.$$  

Since $\Omega$ is a locally compact Hausdorff space, there is a relatively compact open neighborhood $U_i$ of $\phi$ such that $\phi_i \in U_i$ ($i = 1, 2, \ldots, n$). Moreover, there is a compact neighborhood $V_i$ of $\phi$ with $V_i \subseteq U_i$ ($i = 1, 2, \ldots, n$). Since $\hat{A} = C_0(\Omega)$, the algebra of all continuous complex-valued functions on $\Omega$ vanishing at infinity, by [8, Theorem 3E], there exists an $x_i \in \hat{A}$ such that $0 \leq \hat{x}_i \leq 1$, $\hat{x}_i = 1$ on $V_i$, and $\hat{x}_i = 0$ on the complement of $U_i$ ($i = 1, 2, \ldots, n$). Then $\phi(x_i) = 1$ and $\phi_i(x_i) = 0$ ($i = 1, \ldots, n$). Since $\left\| x_1 \cdots x_n \right\| \leq 1$ and $\phi_i(x_1 \cdots x_n) = 0$ ($i = 1, \ldots, n$), by (#) we have that $\left| \phi(x_1 \cdots x_n) \right| < 1$. But $\phi(x_1 \cdots x_n) = \phi(x_1) \cdots \phi(x_n) = 1$; a contradiction. Hence $\phi \notin M$, and the proof is complete.

Let $A$ be a semisimple commutative Banach algebra with carrier space $\Omega$. A function $f$ on $\Omega$ is said to belong locally to $\hat{A}$ at $p \in \Omega$ if there exists a neighborhood $V$ of $p$ and a function $\hat{x} \in \hat{A}$ such that $f|_V = \hat{x}|_V$, where $f|_V$ and $\hat{x}|_V$ denote the restrictions of $f$ and $\hat{x}$ to $V$.

Let $A$ be a commutative $B^*$-algebra and let $J_A(\infty)$ be the set of $x \in A$ such that $\hat{x}$ has a compact support. Since $A$ is also strongly semisimple, by [11, Theorem (2.7.25)] and [5, Théorème (2.9.5) (iii)], we have $\text{cl}(J_A(\infty)) = A$; i.e., $A$ is Tauberian.

For any set $S$ in a Banach algebra $A$, let $S_L$ and $S_R$ denote the left and right annihilators of $S$ in $A$, respectively. $A$ is called an annihilator algebra if, for every closed left ideal $J$ and for every closed right ideal $R$, we have $J = (0)$ if and only if $J = A$ and $R = (0)$ if and only if $R = A$. $A$ is called a dual algebra if $J_{rt} = J$ and $R_{tr} = R$ for all closed left ideals $J$ and all closed right ideals $R$.

Let $H$ be a Hilbert space and $L(H)$ the algebra of all continuous linear operators on $H$ into itself with the usual operator bound norm.
Let $LC(H)$ be the subalgebra of $L(H)$ consisting of all compact operators on $H$ and $\tau c(H)$ the subalgebra of $L(H)$ consisting of all trace class operators on $H$. We are now ready to prove the following theorem.

**Theorem 4.2.** For a commutative $B^*$-algebra $A$, the following statements are equivalent:

1. $A$ is a dual algebra.
2. $\Omega$ is discrete.
3. $M(A) = A''$.
4. For each $F \subseteq A''$, $F$ belongs locally to $\hat{A}$ at each point of $\Omega$.
5. $\pi'(A)$ is an ideal of $A''$.
6. $\pi(A)$ is an ideal of $A^{**}$.
7. The socle of $A$ is strictly dense in $M(A)$.

**Proof.** (1) $\Rightarrow$ (2). Suppose (1) holds. Let $0 \in \Omega$ and let $M = \{a \in A : \phi(a) = 0\}$. Then $M$ is a maximal modular ideal of $A$ and, since $A$ is an annihilator algebra, by [3, Theorem 1], $M = \{x - ex : x \in A\}$, where $e$ is a minimal idempotent of $A$. Clearly $e$ is selfadjoint and $\phi(e) = 1$. It is easy to see that $\phi' = 0$ for all $\phi' \in \Omega$, $\phi' \neq \phi$. Thus $\hat{e}$ is the characteristic function of the set $\{\phi\}$ and since $\hat{e}$ is continuous in the weak topology of $\Omega$, it follows that $\{\phi\}$ is open. Hence $\Omega$ is discrete.

(2) $\Rightarrow$ (1). Suppose (2) holds. Let $M$ be a maximal modular ideal of $A$ and let $\phi$ be the element of $\Omega$ corresponding to $M$. Since $\Omega$ is discrete, the characteristic function of the set $\{\phi\}$ is continuous and hence is the image of an element $e \in A$ by the Gelfand mapping. It is straightforward to show that $M = \{x - ex : x \in A\}$. As $e$ is an idempotent, we have $M_1 \neq (0)$. But, by [5, Théorème (2.9.5) (iii)], each closed ideal of $A$ is the intersection of maximal modular ideals containing it. Hence $A$ is an annihilator algebra and therefore dual by [3, Corollary, Theorem 3].

(1) $\Rightarrow$ (6). Suppose $A$ is dual. (In the argument that follows we may take $A$ to be any dual $B^*$-algebra.) Then, by [7, Lemma 2.3], there exists a family of Hilbert spaces $\{H_\lambda\}$ and $A$ is isometrically $*$-isomorphic to $(\sum LC(H_\lambda))_0$, the $B^*(\infty)$-sum of $\{LC(H_\lambda)\}$. It is easy to verify that $A^*$ is isometrically isomorphic to $(\sum \tau c(H_\lambda))_1$, the $L_1$-direct sum of $\{\tau c(H_\lambda)\}$, and that in turn $A^{**}$ is isometrically isomorphic to the normed full direct sum $\sum L(H_\lambda)$ of $\{L(H_\lambda)\}$. Clearly $(\sum LC(H_\lambda))_0$ is a closed (two-sided) ideal of $\sum L(H_\lambda)$. But the Arens product and the given product coincide on $\sum L(H_\lambda)$ since they coincide on each $L(H_\lambda)$ [11, p. 289]. Hence $\pi(A)$ is a closed (two-sided) ideal of $A^{**}$.
(6)⇒(5). Suppose (6) holds. Let \( F \in A'' \) and let \( \mathcal{F} \) be an isometric extension of \( F \) to all of \( A^* \). Since \( \pi(x) * \mathcal{F} \subseteq \pi(A) \) and

\[
(\pi'(x) \circ F) \mid \Omega = (\pi(x) * \mathcal{F}) \mid \Omega,
\]

\( (\pi'(x) \circ F) \mid \Omega \) is a continuous function on \( \Omega \) vanishing at infinity. Hence \( \pi'(x) \circ F \in \pi'(A) \).

(5)⇒(2). Suppose (5) holds and let \( U \) be a compact subset of \( \Omega \). We claim that \( U \) is finite. Suppose this is not so. Let \( \{ \phi_\gamma \} \) be a net in \( U \) converging to an element \( \phi \) and such that \( \phi_\gamma \not\equiv \phi \) for all \( \gamma \). Let \( M \) be the closed subspace of \( A' \) spanned by the \( \phi_\gamma \). By Lemma 4.1, \( \phi \in M \) and so there exists an \( F \in A'' \) such that \( F(M) = 0 \) and \( F(\phi) \not\equiv 0 \). Let \( x \in A \) be such that \( \phi(x) \not\equiv 0 \). Since \( \phi_\gamma \in M \), \( F \circ \pi'(x)(\phi_\gamma) = F(\phi_\gamma)\phi_\gamma(x) = 0 \). But \( F \circ \pi'(x)(\phi) = F(\phi)\phi(x) \not\equiv 0 \). Hence, since \( \phi_\gamma \to \phi \) in the topology of \( \Omega \), it follows that \( F \circ \pi'(x) \) is not continuous at \( \phi \) and so \( F \circ \pi'(x) \not\in \pi'(A) \), which contradicts the assumption that \( \pi'(A) \) is an ideal of \( A'' \). Hence \( U \) is finite and consequently \( \Omega \) is discrete.

(2)⇒(3). This follows from [2, Theorem, p. 817].

(3)⇒(4). This is [2, Lemma 1].

(4)⇒(3). Suppose (4) holds. Since \( A \) is Tauberian, \( cl(xA) = cl(xJ_A(\infty)) \) for all \( x \in A \) and, since \( A \) has an approximate identity, \( x \in cl(xJ_A(\infty)) \). Hence, by [2, Lemma 3], \( A'' \subseteq M(A) \) and therefore, since \( \pi'(A) \) is an ideal of \( A'' \), then \( \pi'(A) \not\in \pi'(A) \).

(1)⇒(7). This follows from Lemma 2.1 and the fact that a \( B^* \)-algebra is dual if and only if it has a dense socle [7].

5. Duality in a general \( B^* \)-algebra.

**Theorem 5.1.** Let \( A \) be a \( B^* \)-algebra and \( \pi \) the canonical mapping of \( A \) into \( A^{**} \). Then the following statements are equivalent:

(a) \( A \) is a dual algebra.

(b) \( \pi(A) \) is a closed two-sided ideal of \( A^{**} \).

**Proof.** (a)⇒(b). This is given in the proof of (1)⇒(6) of Theorem 4.2.

(b)⇒(a). Suppose (b) holds. Let \( B \) be a maximal commutative \( * \)-subalgebra of \( A \) and \( \pi_1 \) the canonical mapping of \( B \) into \( B^{**} \). For each \( f \in A^* \), let \( f_B = f|_B \), the restriction of \( f \) to \( B \); clearly \( f_B \in B^* \). For each \( F \in B^{**} \), let \( \tilde{f} \) be the linear functional on \( A^* \) defined by

\[
\tilde{F}(f) = F(f_B) \quad (f \in A^*).
\]
Then \( \tilde{F} \in A^{**} \) and it is easy to show that \( F \to \tilde{F} \) is an isometric isomorphism of \( B^{**} \) into \( A^{**} \). Let \( x \in B \). Since

\[
\pi_1(x)(f) = \pi_1(x)(f_B) = f_B(x) = \pi(x)(f) \quad (f \in A^*) ,
\]

we have \( \pi_1(x) = \pi(x) \), for all \( x \in B \), so that \( F \to \tilde{F} \) maps \( \pi_1(B) \) onto \( \pi(B) \).

We shall now show that \( \pi(x) \ast \tilde{F} \in \pi(B) \) for all \( x \in B \) and \( F \in B^{**} \). Let \( y \in B \). Then, for all \( f \in A^* \), we have

\[
((\pi(x) \ast \tilde{F}) \ast \pi(y))(f) = \pi(x)(\tilde{F} \ast (\pi(y) \ast f)) \\
= \tilde{F}((\pi(y) \ast f) \ast x) = F((\pi(y) \ast f \ast x)_B).
\]

Similarly, for all \( f \in A^* \),

\[
(\pi(y) \ast (\pi(x) \ast \tilde{F}))(f) = F((f \ast y \ast x)_B).
\]

But

\[
(\pi(y) \ast f \ast x)_B = (f \ast y \ast x)_B;
\]

in fact, for all \( z \in B \), we have

\[
(\pi(y) \ast f \ast x)_B(z) = f(xzy) = f(yxz) = (f \ast y \ast x)(z) \\
= (f \ast y \ast x)_B(z).
\]

Hence

\[
(\pi(x) \ast \tilde{F}) \ast \pi(y) = \pi(y) \ast (\pi(x) \ast \tilde{F}) \quad (x, y \in B, F \in B^{**}).
\]

Since \( \pi(B) \) is a maximal commutative \(*\)-subalgebra of \( \pi(A) \) and since \( \pi(x) \ast \tilde{F} \in \pi(A) \) (by hypothesis), we have \( \pi(x) \ast \tilde{F} \in \pi(B) \). Now

\[
\pi(x) \ast \tilde{F} = \pi_1(x) \ast \tilde{F} = (\pi_1(x) \ast F)^\sim
\]

and hence \( \pi_1(x) \ast F \in \pi_1(B) \), which shows that \( \pi_1(B) \) is an ideal of \( B^{**} \). Thus \( B \) is dual by Theorem 4.2. Since this is true for every maximal commutative \(*\)-subalgebra \( B \) of \( A \), [10, Theorem 1] shows that \( A \) is a dual algebra.

**References**


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