THE ARENS PRODUCT AND DUALITY IN B^*-ALGEBRAS

B. J. TOMIUK AND PAK-KEN WONG

ABSTRACT. Let A be a B^*-algebra, A^{**} its second conjugate space and π the canonical embedding of A into A^{**}. A^{**} is a B^*-algebra under the Arens product. Our main result states that A is a dual algebra if and only if $\pi(A)$ is a two-sided ideal of A^{**}. Gulick has shown that for a commutative A, $\pi(A)$ is an ideal if and only if the carrier space of A is discrete. As this is equivalent to A being a dual algebra, Gulick's result thus carries over to the general B^*-algebra.

1. **Introduction.** Let A be a (complex) commutative B^*-algebra and let Δ be the set of all nonzero multiplicative linear functionals in A^*, the conjugate space of A. Let A' be the closed span of Δ in A^* and let $A''=A'^*$. Let π' be the embedding of A into A'' given by $\pi'(x)=\pi(x)|A'$, the restriction of $\pi(x)$ to A'. Birtel [2] has introduced a product in A'' under which A'' is a commutative Banach algebra. It follows that the multiplier algebra $M(A)$ can be isometrically embedded in A''. We make use of A'', A^{**} and $M(A)$ to obtain several characterizations of duality for A which we gather together in Theorem 4.2.

2. **The multiplier algebra.** Let A be a semisimple Banach algebra. A mapping T on A into itself is called a multiplier if $(Tx)y=x(Ty)$ for all $x, y\in A$. It is easy to see that T is a bounded linear operator on A and that $M(A)$, the set of all multipliers on A, is a closed commutative subalgebra of the Banach algebra $B(A)$ of all bounded linear operators on A into itself under the usual operator bound norm. $M(A)$ is called the multiplier algebra of A. It is easily shown that $M(A)$ is complete under its strong operator topology (i.e., the topology on $M(A)$ generated by the seminorms $T\to\|Tx\|$, $x\in A$). From now on we shall call the strong operator topology on $M(A)$ the strict topology on $M(A)$ [12]. All algebras and vector spaces under consideration are over the complex field C.

Let A be a semisimple commutative Banach algebra. Then A can be identified as an ideal of $M(A)$. In what follows we shall always consider A as a subalgebra of $M(A)$. A is strictly dense in $M(A)$ if and only if A has an approximate identity (see [12]). Let Ω be the carrier

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space of A and $\hat{A}$ the function algebra on Ω isomorphic to A in the Gelfand theory. Then $M(A)$ can be identified with the set of all complex-valued functions f on Ω such that $f\hat{A}\subset\hat{A}$. The functions f are continuous and if T is the multiplier corresponding to f then the sup norm $\|f\|_\infty \leqq \|T\|$ [12, Theorem 3.1].

LEMMA 2.1. *Let A be a semisimple commutative Banach algebra with an approximate identity and let I be an ideal of A. Then I is dense in A if and only if it is strictly dense in $M(A)$.*

PROOF. Since A has an approximate identity, A is isometrically isomorphic to a subalgebra of $M(A)$. Let $\mathrm{cl}(I)$ and $\mathrm{cl}_s(I)$ denote the norm closure and strict closure of I in $M(A)$, respectively. Suppose $\mathrm{cl}(I)=A$. Since the norm topology is finer than the strict topology on $M(A)$, we have $A=\mathrm{cl}(I)\subset\mathrm{cl}_s(I)$. Since A has an approximate identity, $\mathrm{cl}_s(A)=M(A)$. Hence $\mathrm{cl}_s(I)=M(A)$. Conversely suppose $\mathrm{cl}_s(I)=M(A)$. Let $x\in A$ and let $\{x_\beta\}$ be a net in I converging to x in the strict topology. Then $\lim_\beta\|x_\beta e_\alpha - xe_\alpha\|=0$, for each e_α. Since $x_\beta e_\alpha\in I$, we have $xe_\alpha\in\mathrm{cl}(I)$. Therefore $x\in\mathrm{cl}(I)$ and so $\mathrm{cl}(I)=A$.

3. **The algebras A^{**} and A''.** Let A be a B^*-algebra. It is well known that the two Arens products defined in A^{**} coincide [4, Theorem 7.1]. For completeness we sketch the construction of one of the Arens products in A^{**} which we shall use throughout. We do this in stages as follows. (See [1], [4], [6].) Let $x, y\in A$, $f\in A^*$, F, $G\in A^{**}$.

(i) Define $f*x$ by $(f*x)y=f(xy)$. $f*x\in A^*$.

(ii) Define $G*f$ by $(G*f)x=G(f*x)$. $G*f\in A^*$.

(iii) Define $F*G$ by $(F*G)f=F(G*f)$. $F*G\in A^{**}$.

A^{**} is a B^*-algebra under this product [4, Theorem 7.1] and, when A is embedded canonically in A^{**}, it agrees with the given product on A [1].

Now let A be a commutative B^*-algebra. Following Birtel [2] we define a product on A'' given in stages as follows. Let $x\in A$, $f_i\in\Delta$, F, $G\in A''$, $\alpha_i\in C$. (All sums are finite.)

(1) Let $(\sum\alpha_i f_i)\circ x=\sum\alpha_i f_i(x)f_i$.

(2) Let $F\circ(\sum\alpha_i f_i)=\sum\alpha_i F(f_i)f_i$.

(3) Let $F\circ G$ be given by $F\circ G(\sum\alpha_i f_i)=\sum\alpha_i F(f_i)G(f_i)$.

$F\circ G$ is clearly a continuous linear functional on the span of Δ and therefore can be uniquely extended to a linear functional on all of A'. We denote this extension by the same symbol $F\circ G$. The multiplication thus defined on A'' is commutative and $\|F\circ G\|\leqq\|F\|\,\|G\|$ and π' is an isomorphism of A into A'', taking the product xy into $\pi'(x)\circ\pi'(y)$.

4. **Duality in a commutative B^*-algebra.** Let A be a commutative B^*-algebra with carrier space Ω. Since A has a bounded approximate identity and is a supremum norm algebra, $M(A)$ can be isometrically embedded in A'' [2], [12]. We shall assume in what follows that $M(A)\subset A''$.

LEMMA 4.1. *Let ϕ be an element of Ω and let Φ be a subset of Ω such that $\phi\notin\Phi$. Let M be the closed subspace of A' spanned by the elements of Φ. Then $\phi\notin M$.*

PROOF. Suppose $\phi\in M$. Then there exists $\phi_i\in\Phi$ and $\alpha_i\in C$ $(i=1, 2, \cdots, n)$ such that

$$(\#)\qquad \left\|\phi-\sum_{i=1}^{n}\alpha_i\phi_i\right\|<1.$$

Since Ω is a locally compact Hausdorff space, there is a relatively compact open neighborhood U_i of ϕ such that $\phi_i\notin U_i$ $(i=1, 2, \cdots, n)$. Moreover, there is a compact neighborhood V_i of ϕ with $V_i\subset U_i$ $(i=1, 2, \cdots, n)$. Since $\hat{A}=C_0(\Omega)$, the algebra of all continuous complex-valued functions on Ω vanishing at infinity, by [8, Theorem 3E], there exists an $x_i\in A$ such that $0\leqq\hat{x}_i\leqq 1$, $\hat{x}_i=1$ on V_i and $\hat{x}_i=0$ on the complement of U_i $(i=1, 2, \cdots, n)$. Then $\phi(x_i)=1$ and $\phi_i(x_i)=0$ $(i=1, \cdots, n)$. Since $\|x_1\cdots x_n\|\leqq 1$ and $\phi_i(x_1\cdots x_n)=0$ $(i=1, \cdots, n)$, by (#) we have that $|\phi(x_1\cdots x_n)|<1$. But $\phi(x_1\cdots x_n)=\phi(x_1)\cdots\phi(x_n)=1$; a contradiction. Hence $\phi\notin M$, and the proof is complete.

Let A be a semisimple commutative Banach algebra with carrier space Ω. A function f on Ω is said to belong locally to $\hat{A}$ at $p\in\Omega$ if there exists a neighborhood V of p and a function $\hat{x}\in\hat{A}$ such that $f|V=\hat{x}|V$, where $f|V$ and $\hat{x}|V$ denote the restrictions of f and $\hat{x}$ to V.

Let A be a commutative B^*-algebra and let $J_A(\infty)$ be the set of $x\in A$ such that $\hat{x}$ has a compact support. Since A is also strongly semisimple, by [11, Theorem (2.7.25)] and [5, Théorème (2.9.5) (iii)], we have $\mathrm{cl}(J_A(\infty))=A$; i.e., A is Tauberian.

For any set S in a Banach algebra A, let S_l and S_r denote the left and right annihilators of S in A, respectively. A is called an annihilator algebra if, for every closed left ideal J and for every closed right ideal R, we have $J_r=(0)$ if and only if $J=A$ and $R_l=(0)$ if and only if $R=A$. A is called a dual algebra if $J_{rl}=J$ and $R_{lr}=R$ for all closed left ideals J and all closed right ideals R.

Let H be a Hilbert space and $L(H)$ the algebra of all continuous linear operators on H into itself with the usual operator bound norm.

Let $LC(H)$ be the subalgebra of $L(H)$ consisting of all compact operators on H and $\tau c(H)$ the subalgebra of $L(H)$ consisting of all trace class operators on H. We are now ready to prove the following theorem.

THEOREM 4.2. *For a commutative B^*-algebra A, the following statements are equivalent*:

(1) *A is a dual algebra.*

(2) *Ω is discrete.*

(3) *$M(A)=A''$.*

(4) *For each $F\in A''$, F belongs locally to $\hat{A}$ at each point of Ω.*

(5) *$\pi'(A)$ is an ideal of A''.*

(6) *$\pi(A)$ is an ideal of A^{**}.*

(7) *The socle of A is strictly dense in $M(A)$.*

PROOF. (1)$\Rightarrow$(2). Suppose (1) holds. Let $\phi\in\Omega$ and let $M=\{a\in A:\phi(a)=0\}$. Then M is a maximal modular ideal of A and, since A is an annihilator algebra, by [**3**, Theorem 1], $M=\{x-ex:x\in A\}$, where e is a minimal idempotent of A. Clearly e is selfadjoint and $\phi(e)=1$. It is easy to see that $\phi'(e)=0$ for all $\phi'\in\Omega$, $\phi'\neq\phi$. Thus $\hat{e}$ is the characteristic function of the set $\{\phi\}$ and since $\hat{e}$ is continuous in the weak topology of Ω, it follows that $\{\phi\}$ is open. Hence Ω is discrete.

(2)$\Rightarrow$(1). Suppose (2) holds. Let M be a maximal modular ideal of A and let ϕ be the element of Ω corresponding to M. Since Ω is discrete, the characteristic function of the set $\{\phi\}$ is continuous and hence is the image of an element $e\in A$ by the Gelfand mapping. It is straightforward to show that $M=\{x-ex:x\in A\}$. As e is an idempotent, we have $M_l\neq(0)$. But, by [**5**, Théorème (2.9.5) (iii)], each closed ideal of A is the intersection of maximal modular ideals containing it. Hence A is an annihilator algebra and therefore dual by [**3**, Corollary, Theorem 3].

(1)$\Rightarrow$(6). Suppose A is dual. (In the argument that follows we may take A to be any dual B^*-algebra.) Then, by [**7**, Lemma 2.3], there exists a family of Hilbert spaces $\{H_\lambda\}$ and A is isometrically $*$-isomorphic to $(\sum LC(H_\lambda))_0$, the $B^*(\infty)$-sum of $\{LC(H_\lambda)\}$. It is easy to verify that A^* is isometrically isomorphic to $(\sum\tau c(H_\lambda))_1$, the L_1- direct sum of $\{\tau c(H_\lambda)\}$, and that in turn A^{**} is isometrically isomorphic to the normed full direct sum $\sum L(H_\lambda)$ of $\{L(H_\lambda)\}$. Clearly $(\sum LC(H_\lambda))_0$ is a closed (two-sided) ideal of $\sum L(H_\lambda)$. But the Arens product and the given product coincide on $\sum L(H_\lambda)$ since they coincide on each $L(H_\lambda)$ [**11**, p. 289]. Hence $\pi(A)$ is a closed (two-sided) ideal of A^{**}.

(6)$\Rightarrow$(5). Suppose (6) holds. Let $F\in A''$ and let $\mathfrak{F}$ be an isometric extension of F to all of A^*. Since $\pi(x) * \mathfrak{F}\in\pi(A)$ and

$$(\pi'(x) \circ F) \mid \Omega = (\pi(x) * \mathfrak{F}) \mid \Omega,$$

$(\pi'(x) \circ F)|\Omega$ is a continuous function on Ω vanishing at infinity. Hence $\pi'(x) \circ F\in\pi'(A)$.

(5)$\Rightarrow$(2). Suppose (5) holds and let U be a compact subset of Ω. We claim that U is finite. Suppose this is not so. Let $\{\phi_\gamma\}$ be a net in U converging to an element ϕ and such that $\phi_\gamma\neq\phi$ for all γ. Let M be the closed subspace of A' spanned by the ϕ_γ. By Lemma 4.1, $\phi\notin M$ and so there exists an $F\in A''$ such that $F(M)=(0)$ and $F(\phi)\neq 0$. Let $x\in A$ be such that $\phi(x)\neq 0$. Since $\phi_\gamma\in M$, $F\circ\pi'(x)(\phi_\gamma) = F(\phi_\gamma)\phi_\gamma(x)=0$. But $F\circ\pi'(x)(\phi)=F(\phi)\phi(x)\neq 0$. Hence, since $\phi_\gamma\to\phi$ in the topology of Ω, it follows that $F\circ\pi'(x)$ is not continuous at ϕ and so $F\circ\pi'(x)\notin\pi'(A)$, which contradicts the assumption that $\pi'(A)$ is an ideal of A''. Hence U is finite and consequently Ω is discrete.

(2)$\Rightarrow$(3). This follows from [2, Theorem, p. 817].

(3)$\Rightarrow$(4). This is [2, Lemma 1].

(4)$\Rightarrow$(3). Suppose (4) holds. Since A is Tauberian, $\mathrm{cl}(xA) = \mathrm{cl}(xJ_A(\infty))$ for all $x\in A$ and, since A has an approximate identity, $x\in\mathrm{cl}(xJ_A(\infty))$. Hence, by [2, Lemma 3], $A''\subset M(A)$ and therefore, since $M(A)\subset A''$, $M(A)=A''$.

(3)$\Rightarrow$(5). Since A is an ideal of $M(A)$, so if $M(A)=A''$ then $\pi'(A)$ is an ideal of A''.

(1)$\Leftrightarrow$(7). This follows from Lemma 2.1 and the fact that a B^*-algebra is dual if and only if it has a dense socle [7].

5. Duality in a general B^*-algebra.

THEOREM 5.1. *Let A be a B^*-algebra and π the canonical mapping of A into A^{**}. Then the following statements are equivalent*:

(a) *A is a dual algebra.*

(b) *$\pi(A)$ is a closed two-sided ideal of A^{**}.*

PROOF. (a)$\Rightarrow$(b). This is given in the proof of (1)$\Rightarrow$(6) of Theorem 4.2.

(b)$\Rightarrow$(a). Suppose (b) holds. Let B be a maximal commutative $*$-subalgebra of A and π_1 the canonical mapping of B into B^{**}. For each $f\in A^*$, let $f_B=f|B$, the restriction of f to B; clearly $f_B\in B^*$. For each $F\in B^{**}$, let $\tilde{f}$ be the linear functional on A^* defined by

$$\tilde{F}(f) = F(f_B) \qquad (f \in A^*).$$

Then $\tilde{F}\in A^{**}$ and it is easy to show that $F\to\tilde{F}$ is an isometric isomorphism of B^{**} into A^{**}. Let $x\in B$. Since

$$\widetilde{\pi_1(x)}(f) = \pi_1(x)(f_B) = f_B(x) = \pi(x)(f) \qquad (f \in A^*),$$

we have $\widetilde{\pi_1(x)}=\pi(x)$, for all $x\in B$, so that $F\to\tilde{F}$ maps $\pi_1(B)$ onto $\pi(B)$.

We shall now show that $\pi(x) * \tilde{F}\in\pi(B)$ for all $x\in B$ and $F\in B^{**}$. Let $y\in B$. Then, for all $f\in A^*$, we have

$$\begin{aligned}((\pi(x) * \tilde{F}) * \pi(y))(f) &= \pi(x)(\tilde{F} * (\pi(y) * f)) \\ &= \tilde{F}((\pi(y) * f) * x) = F((\pi(y) * f * x)_B).\end{aligned}$$

Similarly, for all $f\in A^*$,

$$(\pi(y) * (\pi(x) * \tilde{F}))(f) = F((f * y * x)_B).$$

But

$$(\pi(y) * f * x)_B = (f * y * x)_B;$$

in fact, for all $z\in B$, we have

$$\begin{aligned}(\pi(y) * f * x)_B(z) &= f(xzy) = f(yxz) = (f * y * x)(z) \\ &= (f * y * x)(z) = (f * y * x)_B(z).\end{aligned}$$

Hence

$$(\pi(x) * \tilde{F}) * \pi(y) = \pi(y) * (\pi(x) * \tilde{F}) \qquad (x, y \in B, F \in B^{**}).$$

Since $\pi(B)$ is a maximal commutative *-subalgebra of $\pi(A)$ and since $\pi(x) * \tilde{F}\in\pi(A)$ (by hypothesis), we have $\pi(x) * \tilde{F}\in\pi(B)$. Now

$$\pi(x) * \tilde{F} = \widetilde{\pi_1(x)} * \tilde{F} = (\pi_1(x) * F)^{\sim}$$

and hence $\pi_1(x) * F\in\pi_1(B)$, which shows that $\pi_1(B)$ is an ideal of B^{**}. Thus B is dual by Theorem 4.2. Since this is true for every maximal commutative *-subalgebra B of A, [**10**, Theorem 1] shows that A is a dual algebra.

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UNIVERSITY OF OTTAWA, OTTAWA, CANADA