MAXIMAL C^*-SUBALGEBRAS OF A BANACH ALGEBRA

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ABSTRACT. Let A be a complex Banach algebra with identity and let H be its set of hermitain elements. It is shown that $H+iH$ is a C^*-algebra if and only if $h^2 \in H+iH$ whenever $h \in H$; and that every C^*-subalgebra of A is contained in $H+iH$.

Let $(A, \|\cdot\|)$ be a complex Banach algebra with identity e, where $\|e\|=1$. An element h in A is said to be hermitian if $\|e+i\alpha h\|=1+o(\alpha)$ as α approaches 0, α real. Let H be the closed real-linear subspace consisting of all the hermitian elements of A. For g, h in H, let $(g+ih)^*=g-ih$. Then $*$ is conjugate linear and of order two, but not necessarily antimultiplicative.

In 1956 Vidav showed [**7**, p. 9] that if (i) $A=H+iH$ and (ii) $h \in H$ implies that $h^2=u+iv$ where $u, v \in H$ and $uv=vu$, then $(A, \|\cdot\|_0, *)$ is a C^*-algebra for some equivalent norm $\|\cdot\|_0$. Later Berkson [**1**, p. 7] and Glickfeld [**2**, Theorem 2.4, p. 554] proved independently that this equivalent norm $\|\cdot\|_0$ is identical with the original norm, and then Palmer [**5**, p. 539] proved that condition (i) alone is sufficient. Here we show that a much weaker form of condition (ii) implies that $H+iH$ is a C^*-algebra.

THEOREM. *Let A, $\|\cdot\|$, $*$, H be as above. Then $(H+iH, \|\cdot\|, *)$ is a C^*-algebra if and only if $h^2 \in H+iH$ whenever $h \in H$. If B is any subalgebra of A which contains e and is a C^*-algebra under the given norm and some involution, then $B \subseteq H+iH$ and its involution is $*$.*

PROOF. The last sentence and the necessity in the second sentence are obvious. Now assume that $h^2 \in H+iH$ whenever $h \in H$. Since

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$i(gh-hg)\in H$ [7, Lemma 2b, p. 122] and $gh+hg=(g+h)^2-g^2-h^2 \in H+iH$ whenever $g, h\in H$, a calculation shows that if a and b are in $H+iH$ then so are $ab+ba$ and $ab-ba$, whence ab is also. Thus $H+iH$ is an algebra.

To show that $H+iH$ is closed let $g_n+ih_n\to a\in A$, where the g_n, $h_n\in H$. Let $[\cdot,\ \cdot]$ be any semi-inner-product on $(A, \|\cdot\|)$, as defined by Lumer [3, p. 31]. Since the g_n and h_n are in H, $[g_nx, x]$ and $[h_nx, x]$ are real for all x in A and all n [3, p. 39]. Thus

$$\begin{aligned}\sup_{\|x\|=1} |\,[g_nx, x] - \mathrm{Re}[ax, x]\,| &\leqq \sup_{\|x\|=1} |\,[g_nx, x] + i[h_nx, x] - [ax, x]\,| \\ &\leqq \|g_n + ih_n - a\|,\end{aligned}$$

which converges to 0. It follows from a result of Lumer [3, Theorem 5, p. 33] that $\|y\|\leqq 4 \sup \{\,|\,[yx, x]\,| : \|x\| = 1\,\}$, whence

$$\begin{aligned}(1/4)\|g_n - g_m\| &\leqq \sup_{\|x\|=1} |\,[(g_n - g_m)x, x]\,| \\ &\leqq \sup_{\|x\|=1} |\,[g_nx, x] - \mathrm{Re}[ax, x]\,| \\ &\quad + \sup_{\|x\|=1} |\,\mathrm{Re}[ax, x] - [g_mx, x]\,|,\end{aligned}$$

which approaches 0 as n, m increase. Thus $\{g_n\}$ is a Cauchy sequence, hence converges to an element g of H. Similarly, $\{h_n\}$ converges to an element h of H, and $a=\lim_n g_n+ih_n=g+ih$ in $H+iH$. (This last result was also obtained by Berkson [1, Lemma 3.1, p. 6] and Palmer [6, Lemma 3.2, p. 41].)

Since, then, $(H+iH, \|\cdot\|, {}^*)$ is a complex Banach algebra with identity e, $\|e\|=1$, it follows from a previously mentioned result of Palmer [5, p. 539] that it is a C^*-algebra.

One might ask whether $H+iH$ is always an algebra. That this is not the case is shown by an example of Lumer [4, p. 84], who found an operator R on a reflexive Banach space such that R is hermitian but some power R^n is not.

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