ANALYTICITY AND QUASI-ANALYTICITY FOR
ONE-PARAMETER SEMIGROUPS

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Abstract. Suppose that $T$ is a strongly continuous (even at 0) one-parameter semigroup of bounded linear transformations on a real Banach space $S$ and $T$ has generator $A$.

THEOREM A. If \( \limsup_{x \to 0} |T(x) - I| < 2 \) then \( AT(x) \) is bounded for all \( x > 0 \).

Suppose \( \{\delta_q\} \) is a sequence of positive numbers convergent to 0 and each of \( N(q) \), \( q = 1, 2, \ldots \) is an increasing sequence of positive integers. Denote by \( Q \) the collection consisting of (1) all real analytic functions on \((0, \infty)\) and (2) all \( h \) on \((0, \infty)\) for which there is a Banach space \( S \), a member \( p \) of \( S \), a member \( f \) of \( S^* \) and a strongly continuous semigroup \( L \) of bounded linear transformations so that \( h(x) = f[L(x)p] \) for all \( x > 0 \) where \( L \) satisfies \( \limsup_{n \to \infty} |L(\delta_q/n) - I| < 2 \), \( q = 1, 2, \ldots \).

THEOREM B. No two members of \( Q \) agree on an open subset of \((0, \infty)\).

1. Introduction and statement of theorems. Suppose that \( S \) is a real Banach space and \( T \) is a strongly continuous (even at 0) one-parameter semigroup of bounded linear transformations from \( S \) to \( S \). If \( p \) is in \( S \), then the function \( g_p \) so that \( g_p(x) = T(x)p \) for all \( x \geq 0 \) is called a trajectory of \( T \) and, if \( f \) is in \( S^* \), a function \( h \) so that \( h(x) = f(g_p(x)) \) for all \( x > 0 \) will be referred to as a functional of a trajectory of \( T \).

For comparison with the following theorem recall that if \( \lim_{x \to 0} |T(x) - I| = 0 \), then the generator \( A \) of \( T \) is bounded and each functional of a trajectory of \( T \) is analytic \([4, \S 9.4]\).

THEOREM A. If

\[
\limsup_{x \to 0} |T(x) - I| < 2,
\]

then \( AT(x) \) is bounded for all \( x > 0 \).
Corollary. If $T$ satisfies (1) and $T$ can be extended to a group of bounded linear transformations, then $A$ is bounded.

As a consequence of the next theorem, one has that if $T$ satisfies instead of (1) the weaker condition

\[ \limsup_{n \to \infty} |T(3^{-n}) - I| < 2 \]

and each of $h$ and $k$ is a functional of a trajectory of $T$, then $h$ and $k$ do not agree on an open subset of $(0, \infty)$ unless $h = k$. Moreover, if $r$ is an analytic function on $(0, \infty)$ and $h$ is a functional of a trajectory of $T$, then $r$ and $h$ do not agree on an open subset of $(0, \infty)$ unless $r = h$.

Using the first example in §3 one can find a semigroup satisfying (2) so that some functional of one of its trajectories is not analytic.

Before the next theorem is stated some preliminary definitions are given. Denote by $A = \{\delta_q\}_{q=1}^{\infty}$ a sequence of positive numbers converging to 0 and denote by $K = \{N(q)\}_{q=1}^{\infty}$ a sequence each term of which is an increasing sequence of positive integers $(N(q) = \{n_{q,i}\}_{i=1}^{\infty}, q = 1, 2, \ldots )$. Call a strongly continuous semigroup $M$ on a real Banach space $(\Delta, K)$-regular if

\[ \limsup_{j \to \infty} |M(\delta_q/n_{q,i}) - I| < 2, \quad q = 1, 2, \ldots . \]

Denote by $Q(\Delta, K)$ the collection to which $h$ belongs if and only if $h$ is a functional of a trajectory of a $(\Delta, K)$-regular semigroup.

Theorem B. $Q(\Delta, K)$ is a quasi-analytic collection in the sense that no two members of it agree on an open subset of $(0, \infty)$. Moreover, if $h$ is an analytic function on $(0, \infty)$ and a member $k$ of $Q(\Delta, K)$ agrees with $h$ on an open subset of $(0, \infty)$, then $h = k$.

It is known [4, §19.4] that there is a strongly continuous semigroup so that some functional of one of its trajectories is not identically zero on $(0, \infty)$ but is zero on some open subset of $(0, \infty)$. Such a function is not in any of the collections $Q(\Delta, K)$. An example is given in §3 which shows that for $\Delta$ and $K$ properly chosen, $Q(\Delta, K)$ contains a nonanalytic member.

The problem of extending the present development to nonlinear semigroups is illustrated by an example.

Kato [2] has a theorem from which Theorem A follows as a very special case. Williams [9] arrived at the corollary to Theorem A before Theorem A was found. In about 1965 Kendall [3], using [5],
showed that functionals of trajectories of semigroups satisfying (1) formed a quasi-analytic collection. Independently the present writer obtained the weaker result (announced in [6] but never published) that on a given space, the collection of all trajectories of semigroups satisfying (1) form a quasi-analytic collection.

The proof of Theorem A given in this note depends heavily on the following result of Beurling [1] to which this author has had access since early 1968.

Suppose \( f \) is a continuous real-valued function on \([-4, 4]\) and that for some \( \rho \) in \([3/2, 2]\),

\[
\left| \sum_{q=0}^{n} \binom{n}{q} (-1)^{n-q} f(u + q(v - u)/n) \right| \leq \rho^n
\]

if \( u \) and \( v \) are in \([-4, 4]\), \( n = 1, 2, \ldots \). Then \( f \) can be extended analytically to the rhombus with vertices at \( \pm 4, \pm 4i\alpha^2 \) where \( \alpha = (2 - \rho)/4 \).

One can get from Beurling's argument that the following is true: Suppose that \( t \) is a number, \( \delta' > 0 \) and \( G \) is a collection of real-valued continuous functions whose domains include \([t - \delta', t + \delta']\) so that for some \( L > 0 \) and \( \rho \) in \([3/2, 2]\),

\[
\left| \sum_{q=0}^{n} \binom{n}{q} (-1)^{n-q} f(u + q(v - u)/n) \right| \leq L\rho^n
\]

for all \( u, v \) in \([t - \delta', t + \delta']\) and all \( f \) in \( G \), \( n = 0, 1, \ldots \), where \( \sum_{q=0}^{0} \) above is \( f(u) \). Then there exist \( \delta, M > 0 \) so that if \( f \) is in \( G \), then the restriction of \( f \) to \([t - \delta, t + \delta]\) has an analytic extension \( \tilde{f} \) to the closure of the region \( R_\delta(t) (= \{z: |t - z| < \delta\}) \) of the complex plane so that \( |\tilde{f}(z)| \leq M \) for all \( z \) in \( \text{cl}(R_\delta(t)) \).

Theorem B of this note follows from Theorem 2 of [8] which in turn depends on the following slight generalization (Lemma A of [8]) of Lemma 8 of [5]. Suppose that \( u \) and \( v \) are numbers, \( c = (2u/3) + (v/3) \) and \( f \) is a continuous real-valued function whose domain includes \([u, v]\) such that (1) \( f(x) = 0 \) if \( x \) is in \([u, c]\) and (2) if \( y \) is in \((c, v]\) then there is a number \( x \) in \([c, y]\) such that \( f(x) \neq 0 \). Then

\[
\lim_{n \to \infty} \left[ \sum_{s=0}^{n} \binom{n}{s} \sum_{t=0}^{s} \binom{s}{t} (-1)^{s-t} f(u + t(v - u)/n) \right]^{1/n} = 3.
\]

For earlier results concerning the approximation of the identity element by semigroups see [4, §10.7].
2. Proofs.

Proof of Theorem A. If \( f \) is in \( S^* \) and \( p \) is in \( S \), denote by \( h_{p,f} \) the function on \( (0, \infty) \) so that \( h_{p,f}(x) = f[T(x)p] \) for all \( x > 0 \). Denote by each of \( \epsilon \) and \( \delta' \) a positive number so that \( |T(x) - 1| \leq 2 - \epsilon \) if \( 0 \leq x \leq 2\delta' \). Suppose \( t > 0 \) and denote by \( M' \) a positive number so that \( |T(x)| \leq M' \) if \( x \) is in \( J = [t - \delta', t + \delta'] \). Denote by \( G \) the collection of all functions \( h_{p,f} \) so that \( \|p\| \leq 1, |f| \leq 1 \). If each of \( u \) and \( v \) is in \( J \), \( u < v \), \( n \) a nonnegative integer and \( h_{p,f} \) is in \( G \), then

\[
\left| \sum_{q=0}^{n} \binom{n}{q} (-1)^{n-q} h_{p,f}(u + q(v - u)/n) \right|
\]

\[
= \left| \int \sum_{q=0}^{n} \binom{n}{q} (-1)^{n-q} (T(v - u)/n)^q T(u) p \right|
\]

\[
\leq \|f\| |T(u)| \|p\| |T((v - u)/n) - I|^n \leq M'(2 - \epsilon)^n.
\]

By the comment following the statement of Beurling's theorem, there is a positive number \( M \) and a positive number \( \delta \) so that if \( h_{p,f} \) is in \( G \), then the restriction of \( h_{p,f} \) to \( [t - \delta, t + \delta] \) has an analytic extension \( h'_{p,f} \) to \( \text{cl}(R_1(t)) \) such that \( |h'_{p,f}(z)| \leq M \) for all \( z \in \text{cl}(R_1(t)) \). Hence there is a number \( K \) so that \( |h'_{p,f}(x)| \leq K \) for all \( h_{p,f} \) in \( G \) and \( x \) in \( [t - \delta/2, t + \delta/2] = J' \). So, if \( x \) is in \( J' \), \( x \neq t \), and \( h_{p,f} \) is in \( G \), then

\[
|f[(x - t)^{-1}(T(x) - T(t))p]| = |(x - t)^{-1}[h_{p,f}(x) - h_{p,f}(t)]|
\]

\[
= |h'_{p,f}(x_0)| \leq K
\]

for some \( x_0 \) in \( [x, t] \). Hence \( \|(x - t)^{-1}(T(x) - T(t))p\| \leq K \) for all \( x \) in \( J' \) different from \( t \) and all \( p \) in \( S \) such that \( \|p\| \leq 1 \). But this implies that \( \{(x - t)^{-1}(T(x) - T(t)) : 0 < |x - t| < \delta/2 \} \) is bounded. Since \( A \) is densely defined [4, §10.3], then if \( p \) is in the domain of \( A \),

\[
\lim_{z \to t^+} (x - t)^{-1}(T(x) - T(t))p = T(t) \lim_{z \to t^+} (x - t)^{-1}(T(x - t) - I)p = T(t)Ap
\]

and so

\[
\lim_{z \to t^+} (x - t)^{-1}(T(x - t) - I)T(t)p = \lim_{z \to t^+} (x - t)^{-1}(T(x) - T(t))p
\]

exists and is equal to \( AT(t)p \). Hence if \( x_1, x_2, \ldots \) is a decreasing sequence of members of \( J' \) which converges to \( t \) then \( \{(x_n - t)^{-1}(T(x_n) - T(t)) \}_{n=1}^{\infty} \) converges pointwise on a dense subset of \( S \) to \( AT(t) \). From the Banach-Steinhaus theorem it follows that \( \{(x_n - t)^{-1}(T(x_n) - T(t)) \}_{n=1}^{\infty} \) converges pointwise on all of \( S \) to a bounded linear transformation. Such a transformation is a bounded
extension of $AT(t)$. It follows that $AT(t)$ is bounded and since $AT(t)$ is closed it must have domain all of $S$.

**Proof of Theorem B.** Suppose $h$ is a member of $Q(\Delta, K)$. Then for some real Banach space $S$ there is a point $p$ in $S$, a member $f$ of $S^*$ and a strongly continuous semigroup $T$ on $S$ so that $h(x) = f[T(x)p]$ for all $x > 0$. As in the proof of the preceding theorem if $u, v > 0$, $n$ is a positive integer and $r$ is a positive integer $\leq n$, then if $a = |v - u|$ and $M = 1.\mathrm{u.\,b.}_x (T(x))$ |
\[
\sum_{r=0}^{n} \left(\frac{n!}{r!(n-r)!}\right) (-1)^{n-r} h(u + s(v - u)/n) \leq |f| |M||p||T(a/n) - I|^r
\]
and so
\[
D_h(n; u, v) = \sum_{r=0}^{n} \left(\frac{n!}{r!(n-r)!}\right) \sum_{s=0}^{r} \left(\frac{r!}{s!(r-s)!}\right) (-1)^{r-s} h(u + s(v - u)/n)
\]
\[
\leq |f| |M||p| [1 + |T(a/n) - I|]n.
\]
If for some positive integer $q$, $|v - u| = \delta_q$, then
\[
\lim_{n \rightarrow \infty} \sup_{(n \in N(q))} \left[\sum_{r=0}^{n} \left(\frac{n!}{r!(n-r)!}\right) \sum_{s=0}^{r} \left(\frac{r!}{s!(r-s)!}\right) (-1)^{r-s} h(u + s(v - u)/n)\right]^{1/n}
\]
\[
\leq \lim_{n \rightarrow \infty} \sup_{(n \in N(q))} \left[|f| |M|^{1/n} |p|^{1/n} [1 + |T(\delta_q/a) - I|] \right]< 3.
\]

So Theorem B is established. To get an analogous theorem replace the word “real” in each instance by “complex” in the introduction to and proof of Theorem B.
3. **Examples.** In real $l_2$ define $T$ so that if $(a_1, b_1, a_2, b_2, \cdots)$ is in $l_2$, then

$$T(x)(a_1, b_1, a_2, b_2, \cdots) = (a_1 \cos x - b_1 \sin x, a_1 \sin x + b_1 \cos x, a_2 \cos 2x - b_2 \sin 2x, a_2 \sin 2x + b_2 \cos 2x, \cdots)$$

for all $x \geq 0$. Suppose that $0 < \epsilon < 1/2$, $\Delta = \{\delta_q\}_{q=1}^{\infty}$ and $K = \{N(q)\}_{q=1}^{\infty}$, where $n_{q,j}$ are chosen so that

$$\min_{t=0,1,2,\ldots} \left| (p\delta_q/2\pi n_{q,j}) - t \right| \leq \epsilon, \quad q, j = 1, 2, \cdots,$$

holds for infinitely many positive integers $p$. Denote by $k_p$ the sequence so that if $p$ is a positive integer $k_p = 1$ if (4) holds and $k_p = 0$ if (4) does not hold. Denote by $l'_t$ the subspace consisting of all points $(a_1k_1, b_1k_1, a_2k_2, b_2k_2, \cdots)$ so that $(a_1, b_1, a_2, b_2, \cdots)$ is in $l_2$. Denote by $M$ the restriction of $T$ to $l'_t$. Clearly $M$ is strongly continuous on $l'_t$. Some calculation shows that

$$\limsup_{j \to \infty} \left| M(\delta_q/n_{q,j}) - I \right| < (2(1 - \cos 2\pi \epsilon))^{1/2}, \quad q = 1, 2, \cdots,$$

so that all functionals of trajectories of $M$ are in $Q(\Delta, K)$. But, if $P = (a_1k_1, b_1k_1, a_2k_2, b_2k_2, \cdots)$ where $a_p = (p+1)^{-3/4}$, $b_p = 0$, $p = 1, 2, \cdots$, and $h(x) = (M(x)P, P)$ for all $x > 0$, then $h(x) = \sum_{p=1}^{\infty} k_p (p+1)^{-3/2} \cos px$ for all $x > 0$. Since for infinitely many positive integers $p$, $k_p = 1$, it must be that $h$ does not have a continuous second derivative on $(0, \infty)$ and so (see proof of Theorem 2 of [8]) is certainly not analytic on $(0, \infty)$. This shows that the hypothesis of Theorem B does not imply the hypothesis of Theorem A.

For the second example, consider the strongly continuous nonlinear semigroup on $E_1$ given in [7]:

$$T(x) = p - x \quad \text{if } p \geq 1 \quad \text{and} \quad p - x \geq 1, \quad x \geq 0,$$

$$= 1 \quad \text{if } p \geq 1 \quad \text{and} \quad p - x < 1, \quad x > 0,$$

$$= p \quad \text{if } p < 1, \quad x \geq 0.$$

If $p$ is a number define $g_p(x) = T(x)p$ for all $x > 0$. Then if $p > 1$, $g_p \neq g_1$ but $g_p(x) = g_1(x)$ for all $x \geq p - 1$. However, if $x > 0$, the Lipschitz norm of $T(x) - I$, $\left| T(x) - I \right|$, is equal to 1. One expects, nevertheless, interesting generalizations of Theorems A and B to semigroups of nonlinear transformations.
References


8. ———, *Quasi-analytic collections containing Fourier series which are not infinitely differentiable*, J. London Math. Soc. 43 (1968), 612–616. MR 37 #5344.


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