

ANALYTICITY AND QUASI-ANALYTICITY FOR ONE-PARAMETER SEMIGROUPS

J. W. NEUBERGER¹

ABSTRACT. Suppose that T is a strongly continuous (even at 0) one-parameter semigroup of bounded linear transformations on a real Banach space S and T has generator A .

THEOREM A. *If $\limsup_{x \rightarrow 0} |T(x) - I| < 2$ then $AT(x)$ is bounded for all $x > 0$.*

Suppose $\{\delta_q\}_{q=1}^\infty$ is a sequence of positive numbers convergent to 0 and each of $N(q)$, $q=1, 2, \dots$ is an increasing sequence of positive integers. Denote by Q the collection consisting of (1) all real analytic functions on $(0, \infty)$ and (2) all h on $(0, \infty)$ for which there is a Banach space S , a member p of S , a member f of S^* and a strongly continuous semigroup L of bounded linear transformations so that $h(x) = f[L(x)p]$ for all $x > 0$ where L satisfies $\limsup_{n \rightarrow \infty} (\limsup_{(n \in N(q))} |L(\delta_q/n) - I| < 2, q=1, 2, \dots)$.

THEOREM B. *No two members of Q agree on an open subset of $(0, \infty)$.*

1. Introduction and statement of theorems. Suppose that S is a real Banach space and T is a strongly continuous (even at 0) one-parameter semigroup of bounded linear transformations from S to S . If p is in S , then the function g_p so that $g_p(x) = T(x)p$ for all $x \geq 0$ is called a trajectory of T and, if f is in S^* , a function h so that $h(x) = f(g_p(x))$ for all $x > 0$ will be referred to as a functional of a trajectory of T .

For comparison with the following theorem recall that if $\lim_{x \rightarrow 0} |T(x) - I| = 0$, then the generator A of T is bounded and each functional of a trajectory of T is analytic [4, §9.4].

THEOREM A. *If*

$$(1) \quad \limsup_{x \rightarrow 0} |T(x) - I| < 2,$$

then $AT(x)$ is bounded for all $x > 0$.

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COROLLARY. *If T satisfies (1) and T can be extended to a group of bounded linear transformations, then A is bounded.*

As a consequence of the next theorem, one has that if T satisfies instead of (1) the weaker condition

$$(2) \quad \limsup_{n \rightarrow \infty} |T(3^{-n}) - I| < 2$$

and each of h and k is a functional of a trajectory of T , then h and k do not agree on an open subset of $(0, \infty)$ unless $h = k$. Moreover, if r is an analytic function on $(0, \infty)$ and h is a functional of a trajectory of T , then r and h do not agree on an open subset of $(0, \infty)$ unless $r = h$.

Using the first example in §3 one can find a semigroup satisfying (2) so that some functional of one of its trajectories is not analytic.

Before the next theorem is stated some preliminary definitions are given. Denote by $\Delta = \{\delta_q\}_{q=1}^{\infty}$ a sequence of positive numbers converging to 0 and denote by $K = \{N(q)\}_{q=1}^{\infty}$ a sequence each term of which is an increasing sequence of positive integers ($N(q) = \{n_{q,j}\}_{j=1}^{\infty}$, $q = 1, 2, \dots$). Call a strongly continuous semigroup M on a real Banach space (Δ, K) -regular if

$$(3) \quad \limsup_{j \rightarrow \infty} |M(\delta_q/n_{q,j}) - I| < 2, \quad q = 1, 2, \dots$$

Denote by $Q(\Delta, K)$ the collection to which h belongs if and only if h is a functional of a trajectory of a (Δ, K) -regular semigroup.

THEOREM B. *$Q(\Delta, K)$ is a quasi-analytic collection in the sense that no two members of it agree on an open subset of $(0, \infty)$. Moreover, if h is an analytic function on $(0, \infty)$ and a member k of $Q(\Delta, K)$ agrees with h on an open subset of $(0, \infty)$, then $h = k$.*

It is known [4, §19.4] that there is a strongly continuous semigroup so that some functional of one of its trajectories is not identically zero on $(0, \infty)$ but is zero on some open subset of $(0, \infty)$. Such a function is not in any of the collections $Q(\Delta, K)$. An example is given in §3 which shows that for Δ and K properly chosen, $Q(\Delta, K)$ contains a nonanalytic member.

The problem of extending the present development to nonlinear semigroups is illustrated by an example.

Kato [2] has a theorem from which Theorem A follows as a very special case. Williams [9] arrived at the corollary to Theorem A before Theorem A was found. In about 1965 Kendall [3], using [5],

showed that functionals of trajectories of semigroups satisfying (1) formed a quasi-analytic collection. Independently the present writer obtained the weaker result (announced in [6] but never published) that on a given space, the collection of all trajectories of semigroups satisfying (1) form a quasi-analytic collection.

The proof of Theorem A given in this note depends heavily on the following result of Beurling [1] to which this author has had access since early 1968.

Suppose f is a continuous real-valued function on $[-4, 4]$ and that for some ρ in $[3/2, 2)$,

$$\left| \sum_{q=0}^n \binom{n}{q} (-1)^{n-q} f(u + q(v - u)/n) \right| \leq \rho^n$$

if u and v are in $[-4, 4]$, $n = 1, 2, \dots$. Then f can be extended analytically to the rhombus with vertices at $\pm 4, \pm 4i\kappa^2$ where $\kappa = (2 - \rho)/4$.

One can get from Beurling's argument that the following is true: Suppose that t is a number, $\delta' > 0$ and G is a collection of real-valued continuous functions whose domains include $[t - \delta', t + \delta']$ so that for some $L > 0$ and ρ in $[3/2, 2)$,

$$\left| \sum_{q=0}^n \binom{n}{q} (-1)^{n-q} f(u + q(v - u)/n) \right| \leq L\rho^n,$$

for all u, v in $[t - \delta', t + \delta']$ and all f in G , $n = 0, 1, \dots$, where $\sum_{q=0}^n$ above is $f(u)$. Then there exist $\delta, M > 0$ so that if f is in G , then the restriction of f to $[t - \delta, t + \delta]$ has an analytic extension \hat{f} to the closure of the region $R_\delta(t)$ ($= \{z : |t - z| < \delta\}$) of the complex plane so that $|\hat{f}(z)| \leq M$ for all z in $\text{cl}(R_\delta(t))$.

Theorem B of this note follows from Theorem 2 of [8] which in turn depends on the following slight generalization (Lemma A of [8]) of Lemma 8 of [5]. Suppose that u and v are numbers, $c = (2u/3) + (v/3)$ and f is a continuous real-valued function whose domain includes $[u, v]$ such that (1) $f(x) = 0$ if x is in $[u, c]$ and (2) if y is in $(c, v]$ then there is a number x in $[c, y]$ such that $f(x) \neq 0$. Then

$$\lim_{n \rightarrow \infty} \left[\sum_{s=0}^n \binom{n}{s} \left| \sum_{t=0}^s \binom{s}{t} (-1)^{s-t} f(u + t(v - u)/n) \right| \right]^{1/n} = 3.$$

For earlier results concerning the approximation of the identity element by semigroups see [4, §10.7].

2. Proofs.

PROOF OF THEOREM A. If f is in S^* and p is in S , denote by $h_{p,f}$ the function on $(0, \infty)$ so that $h_{p,f}(x) = f[T(x)p]$ for all $x > 0$. Denote by each of ϵ and δ' a positive number so that $|T(x) - I| \leq 2 - \epsilon$ if $0 \leq x \leq 2\delta'$. Suppose $t > 0$ and denote by M' a positive number so that $|T(x)| \leq M'$ if x is in $J = [t - \delta', t + \delta']$. Denote by G the collection of all functions $h_{p,f}$ so that $\|p\| \leq 1, |f| \leq 1$. If each of u and v is in $J, u < v, n$ a nonnegative integer and $h_{p,f}$ is in G , then

$$\begin{aligned} & \left| \sum_{q=0}^n \binom{n}{q} (-1)^{n-q} h_{p,f}(u + q(v - u)/n) \right| \\ &= \left| f \left[\sum_{q=0}^n \binom{n}{q} (-1)^{n-q} (T(v - u)/n)^q T(u)p \right] \right| \\ &\leq |f| \|T(u)\| \|p\| |T((v - u)/n) - I|^n \leq M'(2 - \epsilon)^n. \end{aligned}$$

By the comment following the statement of Beurling's theorem, there is a positive number M and a positive number δ so that if $h_{p,f}$ is in G , then the restriction of $h_{p,f}$ to $[t - \delta, t + \delta]$ has an analytic extension $\hat{h}_{p,f}$ to $\text{cl}(R_\delta(t))$ such that $|\hat{h}_{p,f}(z)| \leq M$ for all z in $\text{cl}(R_\delta(t))$. Hence there is a number K so that $|\hat{h}'_{p,f}(x)| \leq K$ for all $h_{p,f}$ in G and x in $[t - \delta/2, t + \delta/2] = J'$. So, if x is in $J', x \neq t$, and $h_{p,f}$ is in G , then

$$\begin{aligned} |f[(x - t)^{-1}(T(x) - T(t))p]| &= |(x - t)^{-1}[h_{p,f}(x) - h_{p,f}(t)]| \\ &= |\hat{h}'_{p,f}(x_0)| \leq K \end{aligned}$$

for some x_0 in $[x, t]$. Hence $\|(x - t)^{-1}(T(x) - T(t))p\| \leq K$ for all x in J' different from t and all p in S such that $\|p\| \leq 1$. But this implies that $\{|(x - t)^{-1}(T(x) - T(t))| : 0 < |x - t| < \delta/2\}$ is bounded. Since A is densely defined [4, §10.3], then if p is in the domain of A ,

$$\lim_{x \rightarrow t^+} (x - t)^{-1}(T(x) - T(t))p = T(t) \lim_{x \rightarrow t^+} (x - t)^{-1}(T(x - t) - I)p = T(t)Ap$$

and so

$$\lim_{x \rightarrow t^+} (x - t)^{-1}(T(x - t) - I)T(t)p = \lim_{x \rightarrow t^+} (x - t)^{-1}(T(x) - T(t))p$$

exists and is equal to $AT(t)p$. Hence if x_1, x_2, \dots is a decreasing sequence of members of J' which converges to t then $\{(x_n - t)^{-1}(T(x_n) - T(t))\}_{n=1}^\infty$ converges pointwise on a dense subset of S to $AT(t)$. From the Banach-Steinhaus theorem it follows that $\{(x_n - t)^{-1}(T(x_n) - T(t))\}_{n=1}^\infty$ converges pointwise on all of S to a bounded linear transformation. Such a transformation is a bounded

extension of $AT(t)$. It follows that $AT(t)$ is bounded and since $AT(t)$ is closed it must have domain all of S .

PROOF OF THEOREM B. Suppose h is a member of $Q(\Delta, K)$. Then for some real Banach space S there is a point p in S , a member f of S^* and a strongly continuous semigroup T on S so that $h(x) = f[T(x)p]$ for all $x > 0$. As in the proof of the preceding theorem if $u, v > 0$, n is a positive integer and r is a positive integer $\leq n$, then if $a = |v - u|$ and $M = \text{l.u.b.}_{x \in [u, v]} |T(x)|$

$$\left| \sum_{s=0}^r \binom{r}{s} (-1)^{r-s} h(u + s(v-u)/n) \right| \leq |f| M \|p\| |T(a/n) - I|^r$$

and so

$$\begin{aligned} D_h(n; u, v) &\equiv \sum_{r=0}^n \binom{n}{r} \left| \sum_{s=0}^r \binom{r}{s} (-1)^{r-s} h(u + s(v-u)/n) \right| \\ &\leq |f| M \|p\| [1 + |T(a/n) - I|]^n. \end{aligned}$$

If for some positive integer q , $|v - u| = \delta_q$, then

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sup_{(n \in N(q))} [D_h(n; u, v)]^{1/n} \\ \leq \limsup_{n \rightarrow \infty} \sup_{(n \in N(q))} |f|^{1/n} M^{1/n} \|p\|^{1/n} [1 + |T(\delta_q/n) - I|] < 3 \end{aligned}$$

since $\limsup_{j \rightarrow \infty} |T(\delta_q/n_{q,j}) - I| < 2$ and $N(q) = \{n_{q,j}\}_{j=1}^\infty$. Hence according to Theorem 2 of [8], $Q(\Delta, K)$ is a quasi-analytic collection.

Suppose now that k is an analytic function on $(0, \infty)$ and k agrees on some open subset of $(0, \infty)$ with a function h in $Q(\Delta, K)$. Denote $h - k$ by g , denote by q a positive integer and denote by u, v and w three positive numbers so that $v = (2u/3) + (w/3)$, $|u - w| = \delta_q$ and so that $g(x) = 0$ if x is in $[u, v]$ but, if y is in $(v, w]$, then there is a number y' in $[v, y]$ so that $g(y') \neq 0$. By Lemma A of [8], $\lim_{n \rightarrow \infty} D_g(n; u, w)^{1/n} = 3$. This leads to a contradiction since if n is a positive integer, $D_g(n; u, w) = D_{h-k}(n; u, w) \leq D_h(n; u, w) + D_k(n; u, w)$ and so

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sup_{(n \in N(q))} D_g(n; u, w)^{1/n} \\ \leq \max \left\{ \limsup_{n \rightarrow \infty} \sup_{(n \in N(q))} D_h(n; u, w)^{1/n}, \limsup_{n \rightarrow \infty} \sup_{(n \in N(q))} D_k(n; u, w)^{1/n} \right\} < 3. \end{aligned}$$

So Theorem B is established. To get an analogous theorem replace the word "real" in each instance by "complex" in the introduction to and proof of Theorem B.

3. **Examples.** In real l_2 define T so that if $(a_1, b_1, a_2, b_2, \dots)$ is in l_2 , then

$$T(x)(a_1, b_1, a_2, b_2, \dots) = (a_1 \cos x - b_1 \sin x, a_1 \sin x + b_1 \cos x, a_2 \cos 2x - b_2 \sin 2x, a_2 \sin 2x + b_2 \cos 2x, \dots)$$

for all $x \geq 0$. Suppose that $0 < \epsilon < 1/2$, $\Delta = \{\delta_q\}_{q=1}^\infty$ and $K = \{N(q)\}_{q=1}^\infty = \{\{n_{q,j}\}_{j=1}^\infty\}_{q=1}^\infty$ are chosen so that

$$(4) \quad \min_{t=0,1,2,\dots} |(p\delta_q/2\pi n_{q,j}) - t| \leq \epsilon, \quad q, j = 1, 2, \dots,$$

holds for infinitely many positive integers p . Denote by $\{k_p\}_{p=1}^\infty$ the sequence so that if p is a positive integer $k_p = 1$ if (4) holds and $k_p = 0$ if (4) does not hold. Denote by l'_2 the subspace consisting of all points $(a_1 k_1, b_1 k_1, a_2 k_2, b_2 k_2, \dots)$ so that $(a_1, b_1, a_2, b_2, \dots)$ is in l_2 . Denote by M the restriction of T to l'_2 . Clearly M is strongly continuous on l'_2 . Some calculation shows that

$$\limsup_{j \rightarrow \infty} |M(\delta_q/n_{q,j}) - I| < (2(1 - \cos 2\pi\epsilon))^{1/2}, \quad q = 1, 2, \dots,$$

so that all functionals of trajectories of M are in $Q(\Delta, K)$. But, if $P = (a_1 k_1, b_1 k_1, a_2 k_2, b_2 k_2, \dots)$ where $a_p = (p+1)^{-3/4}$, $b_p = 0$, $p = 1, 2, \dots$, and $h(x) = (M(x)P, P)$ for all $x > 0$, then $h(x) = \sum_{p=1}^\infty k_p (p+1)^{-3/2} \cos px$ for all $x > 0$. Since for infinitely many positive integers p , $k_p = 1$, it must be that h does not have a continuous second derivative on $(0, \infty)$ and so (see proof of Theorem 2 of [8]) is certainly not analytic on $(0, \infty)$. This shows that the hypothesis of Theorem B does not imply the hypothesis of Theorem A.

For the second example, consider the strongly continuous nonlinear semigroup on E_1 given in [7]:

$$T(x) = \begin{aligned} & p - x \quad \text{if } p \geq 1 \quad \text{and} \quad p - x \geq 1, x \geq 0, \\ & 1 \quad \text{if } p \geq 1 \quad \text{and} \quad p - x < 1, x > 0, \\ & p \quad \text{if } p < 1, x \geq 0. \end{aligned}$$

If p is a number define $g_p(x) = T(x)p$ for all $x > 0$. Then if $p > 1$, $g_p \neq g_1$ but $g_p(x) = g_1(x)$ for all $x \geq p - 1$. However, if $x > 0$, the Lipschitz norm of $T(x) - I$, $|T(x) - I|$, is equal to 1. One expects, nevertheless, interesting generalizations of Theorems A and B to semigroups of nonlinear transformations.

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EMORY UNIVERSITY, ATLANTA, GEORGIA 30322