ANALYTICITY AND QUASI-ANALYTICITY FOR ONE-PARAMETER SEMIGROUPS

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ABSTRACT. Suppose that T is a strongly continuous (even at 0) one-parameter semigroup of bounded linear transformations on a real Banach space S and T has generator A.

THEOREM A. *If* $\limsup_{x\to 0}|T(x)-I|<2$ *then* $AT(x)$ *is bounded for all* $x>0$.

Suppose $\{\delta_q\}_{q=1}^{\infty}$ is a sequence of positive numbers convergent to 0 and each of $N(q)$, $q=1, 2, \cdots$ is an increasing sequence of positive integers. Denote by Q the collection consisting of (1) all real analytic functions on $(0, \infty)$ and (2) all h on $(0, \infty)$ for which there is a Banach space S, a member p of S, a member f of S^* and a strongly continuous semigroup L of bounded linear transformations so that $h(x)=f[L(x)p]$ for all $x>0$ where L satisfies $\limsup_{n\to\infty\ (n\in N(q))}|L(\delta_q/n)-I|<2, q=1, 2, \cdots$.

THEOREM B. *No two members of Q agree on an open subset of* $(0, \infty)$.

1. Introduction and statement of theorems. Suppose that S is a real Banach space and T is a strongly continuous (even at 0) one-parameter semigroup of bounded linear transformations from S to S. If p is in S, then the function g_p so that $g_p(x)=T(x)p$ for all $x\geqq 0$ is called a trajectory of T and, if f is in S^*, a function h so that $h(x)=f(g_p(x))$ for all $x>0$ will be referred to as a functional of a trajectory of T.

For comparison with the following theorem recall that if $\lim_{x\to 0}|T(x)-I|=0$, then the generator A of T is bounded and each functional of a trajectory of T is analytic [4, §9.4].

THEOREM A. *If*

$$\limsup_{x\to 0}|T(x)-I|<2, \tag{1}$$

then $AT(x)$ *is bounded for all* $x>0$.

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COROLLARY. *If T satisfies* (1) *and T can be extended to a group of bounded linear transformations, then A is bounded.*

As a consequence of the next theorem, one has that if T satisfies instead of (1) the weaker condition

$$\limsup_{n\to\infty} \left| T(3^{-n}) - I \right| < 2 \tag{2}$$

and each of h and k is a functional of a trajectory of T, then h and k do not agree on an open subset of $(0, \infty)$ unless $h=k$. Moreover, if r is an analytic function on $(0, \infty)$ and h is a functional of a trajectory of T, then r and h do not agree on an open subset of $(0, \infty)$ unless $r=h$.

Using the first example in §3 one can find a semigroup satisfying (2) so that some functional of one of its trajectories is not analytic.

Before the next theorem is stated some preliminary definitions are given. Denote by $\Delta=\{\delta_q\}_{q=1}^{\infty}$ a sequence of positive numbers converging to 0 and denote by $K=\{N(q)\}_{q=1}^{\infty}$ a sequence each term of which is an increasing sequence of positive integers ($N(q)=\{n_{q,j}\}_{j=1}^{\infty}$, $q=1, 2, \cdots$). Call a strongly continuous semigroup M on a real Banach space (Δ, K)-regular if

$$\limsup_{j\to\infty} \left| M(\delta_q/n_{q,j}) - I \right| < 2, \qquad q = 1, 2, \cdots. \tag{3}$$

Denote by $Q(\Delta, K)$ the collection to which h belongs if and only if h is a functional of a trajectory of a (Δ, K)-regular semigroup.

THEOREM B. *$Q(\Delta, K)$ is a quasi-analytic collection in the sense that no two members of it agree on an open subset of* $(0, \infty)$. *Moreover, if h is an analytic function on* $(0, \infty)$ *and a member k of $Q(\Delta, K)$ agrees with h on an open subset of* $(0, \infty)$, *then $h=k$.*

It is known [4, §19.4] that there is a strongly continuous semigroup so that some functional of one of its trajectories is not identically zero on $(0, \infty)$ but is zero on some open subset of $(0, \infty)$. Such a function is not in any of the collections $Q(\Delta, K)$. An example is given in §3 which shows that for Δ and K properly chosen, $Q(\Delta, K)$ contains a nonanalytic member.

The problem of extending the present development to nonlinear semigroups is illustrated by an example.

Kato [2] has a theorem from which Theorem A follows as a very special case. Williams [9] arrived at the corollary to Theorem A before Theorem A was found. In about 1965 Kendall [3], using [5], showed that functionals of trajectories of semigroups satisfying (1) formed a quasi-analytic collection. Independently the present writer obtained the weaker result (announced in [**6**] but never published) that on a given space, the collection of all trajectories of semigroups satisfying (1) form a quasi-analytic collection.

The proof of Theorem A given in this note depends heavily on the following result of Beurling [**1**] to which this author has had access since early 1968.

Suppose f is a continuous real-valued function on $[-4, 4]$ and that for some ρ in $[3/2, 2)$,

$$\left| \sum_{q=0}^{n} \binom{n}{q} (-1)^{n-q} f(u + q(v-u)/n) \right| \leqq \rho^n$$

if u and v are in $[-4, 4]$, $n = 1, 2, \cdots$. Then f can be extended analytically to the rhombus with vertices at ± 4, $\pm 4ik\alpha^2$ where $\alpha = (2-\rho)/4$.

One can get from Beurling's argument that the following is true: Suppose that t is a number, $\delta' > 0$ and G is a collection of real-valued continuous functions whose domains include $[t-\delta', t+\delta']$ so that for some $L > 0$ and ρ in $[3/2, 2)$,

$$\left| \sum_{q=0}^{n} \binom{n}{q} (-1)^{n-q} f(u + q(v-u)/n) \right| \leqq L\rho^n,$$

for all u, v in $[t-\delta', t+\delta']$ and all f in G, $n = 0, 1, \cdots$, where $\sum_{q=0}^{0}$ above is $f(u)$. Then there exist δ, $M > 0$ so that if f is in G, then the restriction of f to $[t-\delta, t+\delta]$ has an analytic extension $\hat{f}$ to the closure of the region $R_\delta(t)$ $(= \{z : |t-z| < \delta\})$ of the complex plane so that $|\hat{f}(z)| \leqq M$ for all z in $\mathrm{cl}(R_\delta(t))$.

Theorem B of this note follows from Theorem 2 of [**8**] which in turn depends on the following slight generalization (Lemma A of [**8**]) of Lemma 8 of [**5**]. Suppose that u and v are numbers, $c = (2u/3) + (v/3)$ and f is a continuous real-valued function whose domain includes $[u, v]$ such that (1) $f(x) = 0$ if x is in $[u, c]$ and (2) if y is in $(c, v]$ then there is a number x in $[c, y]$ such that $f(x) \neq 0$. Then

$$\lim_{n \to \infty} \left[\sum_{s=0}^{n} \binom{n}{s} \left| \sum_{t=0}^{s} \binom{s}{t} (-1)^{s-t} f(u + t(v-u)/n) \right| \right]^{1/n} = 3.$$

For earlier results concerning the approximation of the identity element by semigroups see [**4**, §10.7].

2. Proofs.

Proof of Theorem A. If f is in S^* and p is in S, denote by $h_{p,f}$ the function on $(0, \infty)$ so that $h_{p,f}(x)=f[T(x)p]$ for all $x>0$. Denote by each of ϵ and δ' a positive number so that $|T(x)-I|\leqq 2-\epsilon$ if $0\leqq x\leqq 2\delta'$. Suppose $t>0$ and denote by M' a positive number so that $|T(x)|\leqq M'$ if x is in $J=[t-\delta', t+\delta']$. Denote by G the collection of all functions $h_{p,f}$ so that $\|p\|\leqq 1$, $|f|\leqq 1$. If each of u and v is in J, $u<v$, n a nonnegative integer and $h_{p,f}$ is in G, then

$$\begin{aligned}\left|\sum_{q=0}^{n}\binom{n}{q}(-1)^{n-q}h_{p,f}(u+q(v-u)/n)\right| &= \left|f\left[\sum_{q=0}^{n}\binom{n}{q}(-1)^{n-q}(T(v-u)/n))^{q}T(u)p\right]\right| \\ &\leqq |f|\,|T(u)|\,\|p\|\,|T((v-u)/n)-I|^{n}\leqq M'(2-\epsilon)^{n}.\end{aligned}$$

By the comment following the statement of Beurling's theorem, there is a positive number M and a positive number δ so that if $h_{p,f}$ is in G, then the restriction of $h_{p,f}$ to $[t-\delta, t+\delta]$ has an analytic extension $\hat{h}_{p,f}$ to $\mathrm{cl}(R_\delta(t))$ such that $|\hat{h}_{p,f}(z)|\leqq M$ for all z in $\mathrm{cl}(R_\delta(t))$. Hence there is a number K so that $|h'_{p,f}(x)|\leqq K$ for all $h_{p,f}$ in G and x in $[t-\delta/2, t+\delta/2]=J'$. So, if x is in J', $x\neq t$, and $h_{p,f}$ is in G, then

$$\begin{aligned}|f[(x-t)^{-1}(T(x)-T(t))p]| &= |(x-t)^{-1}[h_{p,f}(x)-h_{p,f}(t)]| \\ &= |h'_{p,f}(x_0)|\leqq K\end{aligned}$$

for some x_0 in $[x, t]$. Hence $\|(x-t)^{-1}(T(x)-T(t))p\|\leqq K$ for all x in J' different from t and all p in S such that $\|p\|\leqq 1$. But this implies that $\{|(x-t)^{-1}(T(x)-T(t))| : 0<|x-t|<\delta/2\}$ is bounded. Since A is densely defined [4, §10.3], then if p is in the domain of A,

$$\lim_{x\to t^+}(x-t)^{-1}(T(x)-T(t))p=T(t)\lim_{x\to t^+}(x-t)^{-1}(T(x-t)-I)p=T(t)Ap$$

and so

$$\lim_{x\to t^+}(x-t)^{-1}(T(x-t)-I)T(t)p=\lim_{x\to t^+}(x-t)^{-1}(T(x)-T(t))p$$

exists and is equal to $AT(t)p$. Hence if $x_1, x_2, \cdots$ is a decreasing sequence of members of J' which converges to t then $\{(x_n-t)^{-1}(T(x_n)-T(t))\}_{n=1}^{\infty}$ converges pointwise on a dense subset of S to $AT(t)$. From the Banach-Steinhaus theorem it follows that $\{(x_n-t)^{-1}(T(x_n)-T(t))\}_{n=1}^{\infty}$ converges pointwise on all of S to a bounded linear transformation. Such a transformation is a bounded extension of $AT(t)$. It follows that $AT(t)$ is bounded and since $AT(t)$ is closed it must have domain all of S.

PROOF OF THEOREM B. Suppose h is a member of $Q(\Delta, K)$. Then for some real Banach space S there is a point p in S, a member f of S^* and a strongly continuous semigroup T on S so that $h(x) = f[T(x)p]$ for all $x > 0$. As in the proof of the preceding theorem if $u, v > 0$, n is a positive integer and r is a positive integer $\leqq n$, then if $a = |v-u|$ and $M = \text{l.u.b.}_{x\in[u,v]} |T(x)|$

$$\left| \sum_{s=0}^{r} \binom{r}{s} (-1)^{r-s} h(u+s(v-u)/n) \right| \leqq |f|\, M \|p\| \,|\, T(a/n) - I\,|^r$$

and so

$$\begin{aligned} D_h(n; u, v) &\equiv \sum_{r=0}^{n} \binom{n}{r} \left| \sum_{s=0}^{r} \binom{r}{s} (-1)^{r-s} h(u + s(v-u)/n) \right| \\ &\leqq |f|\, M \|p\| [1 + |\, T(a/n) - I\,|\,]^n. \end{aligned}$$

If for some positive integer q, $|v-u| = \delta_q$, then

$$\begin{aligned} &\limsup_{n\to\infty\ (n\in N(q))} [D_h(n: u, v)]^{1/n} \\ &\qquad \leqq \limsup_{n\to\infty\ (n\in N(q))} |f|^{1/n} M^{1/n} \|p\|^{1/n} [1 + |\, T(\delta_q/n) - I\,|] < 3 \end{aligned}$$

since $\lim \sup_{j\to\infty} |T(\delta_q/n_{q,j}) - I| < 2$ and $N(q) = \{n_{q,j}\}_{j=1}^{\infty}$. Hence according to Theorem 2 of [8], $Q(\Delta, K)$ is a quasi-analytic collection.

Suppose now that k is an analytic function on $(0, \infty)$ and k agrees on some open subset of $(0, \infty)$ with a function h in $Q(\Delta, K)$. Denote $h-k$ by g, denote by q a positive integer and denote by u, v and w three positive numbers so that $v = (2u/3) + (w/3)$, $|u-w| = \delta_q$ and so that $g(x) = 0$ if x is in $[u, v]$ but, if y is in $(v, w]$, then there is a number y' in $[v, y]$ so that $g(y') \neq 0$. By Lemma A of [8], $\lim_{n\to\infty} D_g(n; u, w)^{1/n} = 3$. This leads to a contradiction since if n is a positive integer, $D_g(n: u, w) = D_{h-k}(n; u, w) \leqq D_h(n; u, w) + D_k(n; u, w)$ and so

$$\begin{aligned} &\limsup_{n\to\infty\ (n\in N(q))} D_g(n; u, w)^{1/n} \\ &\leqq \max\left\{ \limsup_{n\to\infty\ (n\in N(q))} D_h(n; u, w)^{1/n}, \limsup_{n\to\infty\ (n\in N(q))} D_k(n; u, w)^{1/n} \right\} < 3. \end{aligned}$$

So Theorem B is established. To get an analagous theorem replace the word "real" in each instance by "complex" in the introduction to and proof of Theorem B.

3. **Examples.** In real l_2 define T so that if $(a_1, b_1, a_2, b_2, \cdots)$ is in l_2, then

$$T(x)(a_1, b_1, a_2, b_2, \cdots)$$
$$= (a_1 \cos x - b_1 \sin x, a_1 \sin x + b_1 \cos x,$$
$$a_2 \cos 2x - b_2 \sin 2x, a_2 \sin 2x + b_2 \cos 2x, \cdots)$$

for all $x \geqq 0$. Suppose that $0 < \epsilon < 1/2$, $\Delta = \{\delta_q\}_{q=1}^{\infty}$ and $K = \{N(q)\}_{q=1}^{\infty} = \{\{n_{q,j}\}_{j=1}^{\infty}\}_{q=1}^{\infty}$ are chosen so that

$$\min_{t=0,1,2,\ldots} |(p\delta_q/2\pi n_{q,j}) - t| \leqq \epsilon, \qquad q, j = 1, 2, \cdots, \tag{4}$$

holds for infinitely many positive integers p. Denote by $\{k_p\}_{p=1}^{\infty}$ the sequence so that if p is a positive integer $k_p = 1$ if (4) holds and $k_p = 0$ if (4) does not hold. Denote by l_2' the subspace consisting of all points $(a_1k_1, b_1k_1, a_2k_2, b_2k_2, \cdots)$ so that $(a_1, b_1, a_2, b_2, \cdots)$ is in l_2. Denote by M the restriction of T to l_2'. Clearly M is strongly continuous on l_2'. Some calculation shows that

$$\limsup_{j \to \infty} |M(\delta_q/n_{q,j}) - I| < (2(1 - \cos 2\pi\epsilon))^{1/2}, \qquad q = 1, 2, \cdots,$$

so that all functionals of trajectories of M are in $Q(\Delta, K)$. But, if $P = (a_1k_1, b_1k_1, a_2k_2, b_2k_2, \cdots)$ where $a_p = (p+1)^{-3/4}$, $b_p = 0$, $p = 1, 2, \cdots$, and $h(x) = (M(x)P, P)$ for all $x > 0$, then $h(x) = \sum_{p=1}^{\infty} k_p(p+1)^{-3/2} \cos px$ for all $x > 0$. Since for infinitely many positive integers p, $k_p = 1$, it must be that h does not have a continuous second derivative on $(0, \infty)$ and so (see proof of Theorem 2 of [8]) is certainly not analytic on $(0, \infty)$. This shows that the hypothesis of Theorem B does not imply the hypothesis of Theorem A.

For the second example, consider the strongly continuous nonlinear semigroup on E_1 given in [7]:

$$\begin{aligned} T(x) &= p - x \quad \text{if } p \geqq 1 \quad \text{and} \quad p - x \geqq 1,\ x \geqq 0, \\ &= 1 \quad \text{if } p \geqq 1 \quad \text{and} \quad p - x < 1,\ x > 0, \\ &= p \quad \text{if } p < 1,\ x \geqq 0. \end{aligned}$$

If p is a number define $g_p(x) = T(x)p$ for all $x > 0$. Then if $p > 1$, $g_p \neq g_1$ but $g_p(x) = g_1(x)$ for all $x \geqq p - 1$. However, if $x > 0$, the Lipshitz norm of $T(x) - I$, $|T(x) - I|$, is equal to 1. One expects, nevertheless, interesting generalizations of Theorems A and B to semigroups of nonlinear transformations.

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