WELL-KNOWN LCA GROUPS CHARACTERIZED BY THEIR CLOSED SUBGROUPS[1]

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ABSTRACT. In this paper we determine (1) the class of all non-discrete LCA groups for which every proper closed subgroup is the kernel of a continuous character of the group, (2) the class of locally compact groups whose closed subgroups are totally ordered by inclusion, and (3) the class of infinite LCA groups whose proper closed subgroups are topologically isomorphic. Since all these determinations involve only the most common LCA groups, we may regard our findings as characterizations of natural classes of these well-known groups.

The program of deriving information about locally compact Abelian (LCA) groups from a knowledge of their closed subgroups has received attention in recent years; see, for example, references [**3**], [**4**], and [**5**]. In this paper we shall state and prove three theorems characterizing some of the most common LCA groups by means of very natural hypotheses upon their closed subgroups.

The LCA groups of which we shall make constant use are the circle T, the additive real numbers R, the integers Z, the cyclic groups $Z(n)$, the quasicyclic groups $Z(p^\infty)$, the p-adic integers J_p and the p-adic numbers F_p. Precise definitions of all these groups may be found in [**1**]; in particular, much detailed information on the groups J_p and F_p may be found in [**1**, §§10 and 25]. Except where explicitly stated, all groups throughout are assumed to be LCA and Hausdorff topological groups. If the group G_1 is topologically isomorphic to the group G_2, we write $G_1 \cong G_2$. The character group of a group G is denoted by $\hat{G}$, and the kernel of a character γ is written as $\ker \gamma$. Finally, if x is an element of a group G, we use the symbol (x) to denote the subgroup of G generated by x; the closure of this subgroup is written as $\overline{(x)}$.

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In [4, Proposition 8] it is shown that the groups J_p and F_p are the only nondiscrete LCA groups for which every proper closed subgroup is open, while in Proposition 6 of the same reference it is demonstrated that the groups T and R are the only nondiscrete LCA groups for which every proper closed subgroup is discrete. In Theorem 1 we show that these four types of groups are the only nondiscrete LCA groups for which every proper closed subgroup is the kernel of a continuous character of the group. We first present a lemma.

LEMMA 1. *Let G be a discrete torsion-free Abelian group such that every proper subgroup is cyclic. Then G is itself cyclic.*

PROOF. G is certainly not divisible (see [1, A.14]). Hence, for some positive integer n, $nG=\{nx:x \text{ in } G\}$ is a proper subgroup of G, so that it is cyclic. Let nx_0 be a generator for nG. Since G is torsion-free, it follows that x_0 is a generator for G, so that G is cyclic.

THEOREM 1. *Let G be a nondiscrete LCA group. Then the following statements are equivalent*:

(1) *Every proper closed subgroup of G has the form* $\ker \gamma$ *for some γ in $\hat{G}$.*

(2) *G is topologically isomorphic to either J_p, F_p (where p is a prime), T or R.*

PROOF. If F is a proper closed subgroup of J_p, then $J_p/F\cong Z(p^n)$ for some positive integer n, and if F is a proper closed subgroup of F_p, then $F_p/F\cong Z(p^\infty)$ (see [1, 10.16a]). If F is a proper closed subgroup of T, then $T/F\cong T$, and if F is a proper closed subgroup of R, then $R/F\cong T$. Hence in all these cases F is the kernel of a continuous character, and so (2)$\Rightarrow$(1).

Conversely, assume (1) and let C be the identity component of G. If C is not trivial, it contains a proper closed subgroup, say H [5, p. 242]. But then H is a proper closed subgroup of G, and so there exists γ in $\hat{G}$ such that $H=\ker \gamma$. Since C is not trivial, it follows that $\gamma(C)=T=\gamma(G)$ so that $G=C+\ker \gamma=C+H=C$. In other words, G is either totally disconnected or connected.

If G is totally disconnected, then $\ker \gamma$ is open in G for every γ in $\hat{G}$ (this follows from [1, 7.7]). Since (1) holds we conclude that every proper closed subgroup of G is open, so $G\cong J_p$ or $G\cong F_p$ by [4, Proposition 7].

Suppose now that G is connected. If H is any proper closed subgroup of G, there is a γ in $\hat{G}$ such that $H=\ker\gamma$, so that $G/H\cong\gamma(G)=T$ [1, 5.29]; hence every proper closed subgroup of $\hat{G}$ must be cyclic. Write $G\cong R^n\times K$, where n is a nonnegative integer and K is compact and connected [1, 9.14], so that $\hat{G}\cong R^n\times D$, where D is discrete and torsion-free. We conclude with the help of Lemma 1 that $\hat{G}\cong R$ or $\hat{G}\cong Z$, so that $G\cong R$ or $G\cong T$. This completes the proof.

REMARK. By dualizing the above theorem we arrive at the statement that the groups $Z(p^\infty)$, F_p, Z and R are the only noncompact LCA groups for which every proper closed subgroup is monothetic.

A familiar series of groups is obtained by a natural process. Let p be a prime, and form the group $Z(p^n)$, where n is a positive integer. The minimal divisible extension [1, A.15] of $Z(p^n)$ is $Z(p^\infty)$. The dual of $Z(p^\infty)$ is J_p [1, 25.2]. The minimal divisible extension of J_p in the sense of [1, 25.32] is F_p. We now formulate a theorem which characterizes these groups by an intrinsic property of their closed subgroups. Note that we do not require that G be Abelian in the hypothesis.

THEOREM 2. *The following are equivalent for a locally compact group* G:

(1) *The closed subgroups of* G *are totally ordered by inclusion.*

(2) G *is topologically isomorphic to either* $Z(p^n)$, $Z(p^\infty)$, J_p *or* F_p, *for some prime* p *and nonnegative integer* n.

PROOF. It follows from [2, p. 4] and [1, 10.16a] that (2) implies (1). Conversely, assume (1). Let x and y be elements of G. The hypothesis (1) assures that either $\overline{(x)}\subseteq\overline{(y)}$ or $\overline{(y)}\subseteq\overline{(x)}$, so that x and y commute; hence G is Abelian. First suppose that G is discrete. Then certainly G is indecomposable, so $G\cong Z(p^n)$ or $G\cong Z(p^\infty)$ by [2, Theorem 10] (note that G cannot be torsion-free, since (1) fails for Z). If G is not discrete, the structure theorem [1, 24.30] shows that G has a compact open subgroup H, since R does not satisfy (1). Then H must satisfy (1), so that its discrete dual also satisfies (1). Thus, by the above, $\hat{H}\cong Z(p^n)$ or $\hat{H}\cong Z(p^\infty)$. The former is impossible, since G is assumed nondiscrete. Thus $H\cong J_p$. It now follows from (1) and [1, 10.16a] that every proper closed subgroup of G is open, so that $G\cong J_p$ or $G\cong F_p$ by [4, Proposition 7]. This completes the proof.

REMARK. With added effort it may be shown that the groups of the above theorem are also the only nondiscrete LCA groups for which the subgroups of the form $\ker\gamma$ for γ in $\hat{G}$ are totally ordered by inclusion.

The p-adic numbers F_p bear much the same relationship to the p-adic integers J_p as the group R does to Z. Indeed, every proper closed subgroup of F_p (respectively R) is topologically isomorphic to J_p (respectively Z). In Theorem 3 we prescribe a simple property of the closed subgroups of a group G which characterizes these four types of groups. We first present a lemma, analogous to Lemma 1.

LEMMA 2. *Let G be an infinite discrete Abelian group such that all its proper quotients are isomorphic. Then $G \cong Z(p^\infty)$ for some prime p.*

PROOF. By the structure theorem for divisible groups [1, A.14], it suffices to show that G is divisible. If not, then $nG \subsetneq G$ for some positive integer n. The quotient G/nG is then a group of bounded order, so that it is a direct sum of finite cyclic groups [2, Theorem 6]. But then the hypothesis on G ensures that G/nG is cyclic of prime order p, and therefore every proper quotient of G is cyclic of prime order p. This implies the contradiction that G is finite, thus completing the proof.

THEOREM 3. *Let G be an infinite LCA group. Then the following are equivalent*:

(1) *All proper closed subgroups of G are topologically isomorphic.*

(2) *G is topologically isomorphic to either Z, R, J_p or F_p, where p is a prime.*

PROOF. We have already observed that (2) implies (1). Conversely, if (1) holds, we conclude from the structure theorem [1, 24.30] that either $G \cong R$ or G has a compact open subgroup. In the latter case, either G is discrete or it is not. If G is discrete, it must be torsion-free, since otherwise every proper subgroup of G would be finite, and so $G \cong Z(p^\infty)$ [4, Corollary 4], for which (1) fails. But if G is torsion-free, then Lemma 1 implies that $G \cong Z$. If G is not discrete, then it has a proper compact open subgroup, whence by (1) all proper closed subgroups are compact. This implies that either $G \cong F_p$ or G is compact, by [4, Corollary 9]. If G is compact, then $\hat{G}$ satisfies the hypothesis of Lemma 2, so that $\hat{G} \cong Z(p^\infty)$ for some prime p. Hence $G \cong J_p$, thus completing the proof.

REMARK. It follows directly from the theorem above that the groups Z and J_p are the only infinite LCA groups which are topologically isomorphic to all their proper closed subgroups.

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