AN EXTENSION OF HARTOGS' THEOREM FOR DOMAINS WHOSE BOUNDARY IS NOT SMOOTH

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ABSTRACT. In this note we obtain an extension of Hartogs' theorem on analytic continuation inside a bounded domain in $\mathbf{C}^n$ which requires no assumption on the smoothness of the boundary. The standard proof of Hartogs' theorem applies with the minor change of using a Whitney function as the boundary data.

Let Ω be a bounded domain in $\mathbf{C}^n$, $n>1$, whose complement is connected. According to Hartogs' theorem, if $\partial\Omega$ is a surface of class C^2 and if $u\in C^2(\partial\Omega)$ satisfies the tangential Cauchy-Riemann equations on $\partial\Omega$, then u is the restriction to $\partial\Omega$ of a function $U\in C^1(\bar{\Omega})$ analytic inside Ω. In this note we observe that the smoothness condition $\partial\Omega\in C^2$ may be discarded in favor of the very mild hypothesis that Ω equals the interior of its closure, providing the boundary data are taken to be a Whitney function of class C^2 which satisfies all the Cauchy-Riemann equations on $\partial\Omega$. We remark that an ordinary function u which satisfies the tangential Cauchy-Riemann equations on a smooth surface S may be assigned a normal derivative to form a Whitney function on S which satisfies all the Cauchy-Riemann equations.

We follow closely the proof of Hartogs' theorem in §2.3 of Hörmander [1]. If K is a compact set in $\mathbf{C}^n$, let $\mathcal{E}^m(K)$ be the space of (complex-valued) Whitney functions on K of class C^m. (Chapter I of Malgrange [2] contains a concise introduction to Whitney functions.) We use the notation $\nabla=(\partial/\partial z_1, \cdots, \partial/\partial \bar{z}_1, \cdots)$ for the gradient operator.

THEOREM. *Let Ω be a bounded domain in $\mathbf{C}^n$, $n>1$, such that $\Omega=\operatorname{Int}(\bar{\Omega})$ and $\Omega'=\mathbf{C}^n\sim\bar{\Omega}$ is connected. If $u\in\mathcal{E}^2(\partial\Omega)$ and if $\bar{\partial}u=\nabla(\bar{\partial}u)=0$, then u may be extended to a function $U\in\mathcal{E}^1(\bar{\Omega})$ analytic in Ω.*

PROOF. By the Whitney extension theorem, there is a function $v\in C^2(\mathbf{C}^n)$ which is an extension of u. Define a differential form of type (0, 1),

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$$f = \bar{\partial} v \text{ in } \Omega, \qquad f = 0 \text{ on } \mathbf{C}^n \sim \Omega.$$

Then $f \in C^1(\mathbf{C}^n)$, $\bar{\partial} f = 0$, and f has compact support. By Theorem 2.3.1 of Hörmander, there is a function $w \in C_c^1(\mathbf{C}^n)$ such that $\bar{\partial} w = f$. We note that $\partial\Omega' = \partial\Omega$, because $\Omega = \text{Int}(\bar{\Omega})$; it follows by unique continuation that $w | \partial\Omega = 0$. Of course the function $U = v - w$ provides the desired analytic continuation of u.

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