ANALYTICITY AND CONTINUATION OF CERTAIN FUNCTIONS OF TWO COMPLEX VARIABLES

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Abstract. This paper shows that the satisfaction of a certain quadratic relation is a sufficient condition that a continuous, symmetric function of two complex variables on a domain be analytic and be continuable to a particular larger domain. This quadratic relation is of the same type as that involved in the Grunsky inequalities.

In proving a generalization of the Grunsky inequalities, Bergman and Schiffer [3] announced a theorem on analytic continuation of a function of two complex variables. In extending the Grunsky inequalities in another way, Alenicyn [1] found this theorem on continuation useful. The purpose of this note is to strengthen the Bergman-Schiffer theorem to be more natural for both applications and to provide a proof that is more direct than the formal computation in the original proof.

Suppose $\mathcal{D}$ and $\mathcal{G}$ are bounded domains, and $\mathcal{G}$ is contained in $\mathcal{D}$. Let $\int_{\mathcal{D}} dA_z$ denote area integration as $z$ ranges over $\mathcal{D}$. Let $K_{\mathcal{D}}(z, \xi)$ be the Bergman kernel function [2] for the domain $\mathcal{D}$.

**Theorem.** If $V(z, \xi)$ is a symmetric, continuous, complex-valued function on $\mathcal{G} \times \mathcal{G}$, and

\[
\left| \int_{\mathcal{G}} \int_{\mathcal{G}} V(z, \xi) \overline{\phi(z)} \phi(\xi) dA_z dA_\xi \right| \leq \int_{\mathcal{G}} \int_{\mathcal{G}} K_{\mathcal{D}}(z, \xi) \overline{\phi(z)} \phi(\xi) dA_z dA_\xi
\]

for all continuous, complex-valued function $\phi$ with compact support in $\mathcal{G}$, then $V(z, \xi)$ is analytic in $\mathcal{G} \times \mathcal{G}$ and can be continued onto $\mathcal{D} \times \mathcal{D}$.

**Proof.** Let $G$ be a subdomain of $\mathcal{G}$ such that the closure $\overline{G}$ is contained in $\mathcal{G}$. There exists a complete orthonormal system of analytic functions $\{\phi_n\}_{n=1}^\infty$ on $\mathcal{D}$, which is also orthogonal on $G$, [2]. Let

\[
k_n^2 = \int_{\mathcal{G}} \phi_n(z) \overline{\phi_n(z)} dA_z \quad \text{for } n = 1, 2, \ldots
\]

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Then \( \{ \phi_n(z)/k_n \}_{n=1}^{\infty} \) is an orthonormal system on \( G \), but is not necessarily a complete system.

Let \( \alpha_{nm} \) be defined by

\[
(2) \quad k_n k_m \alpha_{nm} = \int_G \int_G V(z, \xi) \overline{\phi_n(z)} \phi_m(\xi) \, dA_z dA_\xi.
\]

Since \( V(z, \xi) \) is continuous on \( \overline{G} \times \overline{G} \), \( \int_G \int_G |V(z, \xi)|^2 \, dA_z dA_\xi < \infty \). Then by the usual argument, [2]

\[
\sum_{n,m=1}^{\infty} \alpha_{nm} \phi_n(z) \phi_m(\xi)
\]

converges uniformly on compact subsets of \( G \). It is now shown that the series converges to \( V(z, \xi) \).

Suppose \( \Gamma_1(z) \) is a continuous function on \( G \), has \( \int_G |\Gamma_1(z)|^2 \, dA_z < \infty \), and is orthogonal to \( \phi_n \) on \( G \) for \( n = 1, 2, \ldots \). Let \( \Gamma_2(z) \) be any continuous function on \( G \), and \( \lambda \) be a real number. By using the symmetry of \( V(z, \xi) \),

\[
\left| \int_G \int_G V(z, \xi) [\Gamma_1(z) + \lambda \Gamma_2(z)][\Gamma_1(\xi) + \lambda \Gamma_2(\xi)] \, dA_z dA_\xi \right|
\]

\[
= \left| \int_G \int_G V(z, \xi) \overline{\Gamma_1(z)} \Gamma_1(\xi) \, dA_z dA_\xi \right|
\]

\[
+ 2\lambda \int_G \int_G V(z, \xi) \overline{\Gamma_1(z)} \Gamma_2(\xi) \, dA_z dA_\xi
\]

\[
+ \lambda^2 \int_G \int_G V(z, \xi) \overline{\Gamma_2(z)} \Gamma_2(\xi) \, dA_z dA_\xi
\]

on the other hand, by (1) and the orthogonality of \( \Gamma_1 \) to \( \phi_n \) on \( G \)

\[
\leq \lambda^2 \int_G \int_G K_D(z, \xi) \overline{\Gamma_2(z)} \Gamma_2(\xi) \, dA_z dA_\xi \quad \text{for all real} \ \lambda.
\]

Thus

\[
\int_G \int_G V(z, \xi) \overline{\Gamma_1(z)} \Gamma_1(\xi) \, dA_z dA_\xi = 0
\]

and

\[
(3) \quad \int_G \int_G V(z, \xi) \overline{\Gamma_1(z)} \Gamma_2(\xi) \, dA_z dA_\xi = 0.
\]

Letting
\[ \Gamma_2(\xi) = \int_G V(z, \xi) \overline{\Gamma_1(z)} \, dA_z, \]

by (3)

\[ \int_G \left\{ \int_G V(z, \xi) \overline{\Gamma_1(z)} \left[ \int_G V(z, \xi) \overline{\Gamma_1(z)} \, dA_z \right] \, dA_t \right\} \, dA_t = 0, \]

\[ \int_G \int_G V(z, \xi) \overline{\Gamma_1(z)} \, dA_z \, dA_t = 0. \]

Hence

(4) \[ \int_G V(z, \xi) \overline{\Gamma_1(z)} \, dA_z = 0 \quad \text{for all } \xi \text{ in } G, \]

for all functions \( \Gamma_1(z) \) that are continuous on \( G \), have \( \int_G |\Gamma_1(z)|^2 \, dA_z < \infty \) and are orthogonal to \( \phi_n(z) \) on \( G \) for \( n = 1, 2, \ldots \).

Let \( \delta_k(z-z_0) \) be the \( k \)-th continuous approximation to the delta function at \( z_0 \) such that \( \delta_k(z-z_0) = 0 \) for all \( z \) in \( G \) with \( |z-z_0| > 1/k \).

Then \( \delta_k(z-z_0) \) can be expressed by

\[ \Psi_k(z) + \sum_{n=1}^{\infty} b_n(k) \phi_n(z) \]

for \( z \) in \( G \), where \( \Psi_k(z) \) is orthogonal to \( \phi_n(z) \) for \( n = 1, 2, \ldots \), on \( G \), has \( \int_G |\Psi_k(z)|^2 \, dA_z < \infty \) and is continuous on \( G \). By (4), the definition of \( \alpha_{nm} \),

\[ \int_G \int_G \left[ V(z, \xi) - \sum_{n,m=1}^{\infty} \alpha_{nm} \phi_n(z) \phi_m(\xi) \right] \left[ \Psi_k(z) + \sum_{n=1}^{\infty} b_n(k) \phi_n(z) \right] \left[ \Psi_k(\xi) + \sum_{m=1}^{\infty} b_m(k) \phi_m(\xi) \right] \, dA_z \, dA_t = 0. \]

If \( z_0 \) is in \( G \), taking \( \lim_{k \to \infty} \) yields

(5) \[ V(z_0, z_0) - \sum_{n,m=1}^{\infty} \alpha_{nm} \phi_n(z_0) \phi_m(z_0) = 0. \]

A similar computation using \( \delta_k(z-z_0) + \delta_k(z-z_1) \) for the test function yields

\[ V(z_0, z_0) - \sum_{n,m=1}^{\infty} \alpha_{nm} \phi_n(z_0) \phi_m(z_0) + V(z_0, z_1) - \sum_{n,m=1}^{\infty} \alpha_{nm} \phi_n(z_0) \phi_m(z_1) \]

\[ + V(z_1, z_0) - \sum_{n,m=1}^{\infty} \alpha_{nm} \phi_n(z_1) \phi_m(z_0) + V(z_1, z_1) - \sum_{n,m=1}^{\infty} \alpha_{nm} \phi_n(z_1) \phi_m(z_1) = 0. \]
By (5) and the symmetry of $V$ and thus of $\alpha_{nm}$,

$$V(z, \xi) = \sum_{n,m=1}^{\infty} \alpha_{nm} \phi_n(z) \phi_m(\xi)$$

for all $z$ and $\xi$ in $G$.

Hence $V$ is analytic on $G \times G$.

It is now shown that the series (6) converge for $z$ and $\xi$ in $D$.

Let $\theta_k(z) + \sum_{n=1}^{\infty} C_n^{(k)} \phi_n(z)$ be the representation of the $k$th continuous approximation to the delta function at $z_0$ where the representation holds for $z$ in $D$, and $\theta_k(z)$ is orthogonal to $\phi_n(z)$ on $D$ for $n = 1, 2, \ldots$.

$$\int_D \int_D \left[ \sum_{n,m=1}^{L} \alpha_{nm} \phi_n(z) \phi_m(\xi) \right] \left[ \theta_k(z) + \sum_{n=1}^{\infty} C_n^{(k)} \phi_n(z) \right] dA_z dA_{\xi}$$

$$= \int_D \int_D \left[ \sum_{n,m=1}^{L} \alpha_{nm} \phi_n(z) \phi_m(\xi) \right] \left[ \sum_{n=1}^{L} \frac{C_n^{(k)}}{k_n} \phi_n(z) \right] dA_z dA_{\xi}$$

$$= \int_D \int_D \left[ \sum_{n,m=1}^{L} \alpha_{nm} \phi_n(z) \phi_m(\xi) \right] \left[ \sum_{n=1}^{L} \frac{C_n^{(k)}}{k_m} \phi_m(\xi) \right] dA_z dA_{\xi}$$

$$\leq \int_D \int_D K_D(z, \xi) \left[ \sum_{n=1}^{L} \frac{C_n^{(k)}}{k_n} \phi_n(z) \right] \left[ \sum_{m=1}^{L} \frac{C_m^{(k)}}{k_m} \phi_m(\xi) \right] dA_z dA_{\xi}$$

$$\leq \int_D \int_D K_D(z, \xi) \left[ \theta_k(z) + \sum_{n=1}^{\infty} C_n^{(k)} \phi_n(z) \right] \left[ \theta_k(\xi) + \sum_{m=1}^{\infty} C_m^{(k)} \phi_m(\xi) \right] dA_z dA_{\xi}$$
taking the $\lim_{k \to \infty}$

\[
(7) \quad K_\omega(z_0, \bar{z}_0) \geq \left| \sum_{n, m=1}^{L} \alpha_{nm}\phi_n(z_0)\phi_m(z_0) \right| \quad \text{for all } L.
\]

A similar computation using a representation of an approximation of the delta function at $z_0$ plus the delta function at $z_1$ and utilizing (7) yields

\[
K_\omega(z_0, \bar{z}_0) + \Re K_\omega(z_0, \bar{z}_1) + K_\omega(z_1, \bar{z}_1) \geq \left| \sum_{n, m=1}^{L} \alpha_{nm}\phi_n(z_0)\phi_m(z_1) \right|.
\]

Hence $\{ \sum_{n,m=1}^{L} \alpha_{nm}\phi_n(z)\phi_m(\zeta) \}_{L=1}^{\infty}$ is a normal family on $\mathbb{D} \times \mathbb{D}$. Since it converges to $V(z, \zeta)$ on $G \times G$, $\sum_{n,m=1}^{\infty} \alpha_{nm}\phi_n(z)\phi_m(\zeta)$ must converge to an analytic function on $\mathbb{D} \times \mathbb{D}$ that is a continuation of $V(z, \zeta)$.

**Corollary.** If $V(z, \zeta)$ is a symmetric, continuous, complex-valued function on $G \times G$ and

\[
\sum_{n=1}^{L} \alpha_n \alpha_m V(z_n, z_m) \leq \sum_{n=1}^{L} \sum_{m=1}^{L} \alpha_n \alpha_m K_\omega(z_n, \bar{z}_m)
\]

for all complex vectors $(\alpha_1, \alpha_2, \cdots)$, and $(z_1, z_2, \cdots)$ with all $z_n$ in $G$, then $V(z, \zeta)$ is analytic in $G \times G$ and can be continued onto $\mathbb{D} \times \mathbb{D}$.

**References**


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