ANALYTICITY AND CONTINUATION OF CERTAIN
FUNCTIONS OF TWO COMPLEX VARIABLES1

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Abstract. This paper shows that the satisfaction of a certain
quadratic relation is a sufficient condition that a continuous, sym-
metric function of two complex variables on a domain be analytic
and be continuable to a particular larger domain. This quadratic
relation is of the same type as that involved in the Grunsky in-
equalities.

In proving a generalization of the Grunsky inequalities, Bergman
and Schiffer [3] announced a theorem on analytic continuation of a
function of two complex variables. In extending the Grunsky in-
equalities in another way, Alenicyn [1] found this theorem on con-
tinuation useful. The purpose of this note is to strengthen the Berg-
man-Schiffer theorem to be more natural for both applications and to
provide a proof that is more direct than the formal computation in
the original proof.

Suppose $\mathcal{D}$ and $\mathcal{G}$ are bounded domains, and $\mathcal{G}$ is contained in $\mathcal{D}$. Let $\int_{\mathcal{D}} dA_z$ denote area integration as $z$ ranges over $\mathcal{D}$. Let $K_{\mathcal{D}}(z, \xi)$ be the Bergman kernel function [2] for the domain $\mathcal{D}$.

**Theorem.** If $V(z, \xi)$ is a symmetric, continuous, complex-valued
function on $\mathcal{G} \times \mathcal{G}$, and

\[
\left| \int_{\mathcal{G}} \int_{\mathcal{G}} V(z, \xi) \overline{\phi(z)} \phi(\xi) dA_z dA_\xi \right| \leq \int_{\mathcal{G}} \int_{\mathcal{G}} K_{\mathcal{D}}(z, \xi) \overline{\phi(z)} \phi(\xi) dA_z dA_\xi
\]

for all continuous, complex-valued function $\phi$ with compact support in $\mathcal{G}$, then $V(z, \xi)$ is analytic in $\mathcal{G} \times \mathcal{G}$ and can be continued onto $\mathcal{D} \times \mathcal{D}$.

**Proof.** Let $G$ be a subdomain of $\mathcal{G}$ such that the closure $\overline{G}$ is con-
tained in $\mathcal{G}$. There exists a complete orthonormal system of analytic
functions $\{\phi_n\}_{n=1}^\infty$ on $\mathcal{D}$, which is also orthogonal on $G$, [2]. Let

\[
k_n^2 = \int_{\mathcal{G}} \phi_n(z) \overline{\phi_n(z)} dA_z \quad \text{for } n = 1, 2, \ldots
\]

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Then \( \{\phi_n(z)/k_n\}_{n=1}^\infty \) is an orthonormal system on \( G \), but is not necessarily a complete system.

Let \( a_{nm} \) be defined by

\[
(2) \quad k_n k_m a_{nm} = \int_G \int_G V(z, \zeta) \overline{\phi_n(z)} \phi_m(\zeta) dA_z dA_\zeta.
\]

Since \( V(z, \zeta) \) is continuous on \( \overline{G} \times \overline{G} \), \( \int_G \int_G |V(z, \zeta)|^2 dA_z dA_\zeta < \infty \). Then by the usual argument, [2]

\[
\sum_{n,m=1}^\infty a_{nm} \phi_n(z) \phi_m(\zeta)
\]

converges uniformly on compact subsets of \( G \). It is now shown that the series converges to \( V(z, \zeta) \).

Suppose \( \Gamma_1(z) \) is a continuous function on \( G \), has \( \int_G |\Gamma_1(z)|^2 dA_z < \infty \), and is orthogonal to \( \phi_n \) on \( G \) for \( n = 1, 2, \ldots \). Let \( \Gamma_2(z) \) be any continuous function on \( \overline{G} \), and \( \lambda \) be a real number. By using the symmetry of \( V(z, \zeta) \),

\[
\left| \int_G \int_G V(z, \zeta) [\Gamma_1(z) + \lambda \Gamma_2(z)] [\Gamma_1(\zeta) + \lambda \Gamma_2(\zeta)] dA_z dA_\zeta \right|
\]

\[
= \left| \int_G \int_G V(z, \zeta) \overline{\Gamma_1(z)} \Gamma_1(\zeta) dA_z dA_\zeta \right|
\]

\[
+ 2\lambda \int_G \int_G V(z, \zeta) \overline{\Gamma_1(z)} \Gamma_2(\zeta) dA_z dA_\zeta
\]

\[
+ \lambda^2 \int_G \int_G V(z, \zeta) \overline{\Gamma_2(z)} \Gamma_2(\zeta) dA_z dA_\zeta
\]

on the other hand, by (1) and the orthogonality of \( \Gamma_1 \) to \( \phi_n \) on \( G \)

\[
\leq \lambda^2 \int_G \int_G K_{\phi}(z, \zeta) \overline{\Gamma_2(z)} \Gamma_2(\zeta) dA_z dA_\zeta
\]

for all real \( \lambda \).

Thus

\[
\int_G \int_G V(z, \zeta) \overline{\Gamma_1(z)} \Gamma_1(\zeta) dA_z dA_\zeta = 0
\]

and

\[
\int_G \int_G V(z, \zeta) \overline{\Gamma_1(z)} \Gamma_2(\zeta) dA_z dA_\zeta = 0.
\]

Letting
\[
\Gamma_1(\xi) = \int_G V(z, \xi) \overline{\Gamma_1(z)} \, dA_z,
\]
by (3)
\[
\int_G \left\{ \int_G V(z, \xi) \overline{\Gamma_1(z)} \left[ \int_G V(z, \xi) \overline{\Gamma_1(z)} \, dA_z \right] dA_z \right\} \, dA_t = 0,
\]
\[
\int_G \left| \int_G V(z, \xi) \overline{\Gamma_1(z)} \, dA_z \right|^2 \, dA_t = 0.
\]
Hence
\[
(4) \quad \int_G V(z, \xi) \overline{\Gamma_1(z)} \, dA_z = 0 \quad \text{for all } \xi \text{ in } G,
\]
for all functions \( \Gamma_1(z) \) that are continuous on \( G \), have \( \int_G \left| \Gamma_1(z) \right|^2 \, dA_z < \infty \) and are orthogonal to \( \phi_n(z) \) on \( G \) for \( n = 1, 2, \ldots \).

Let \( \delta_k(z - z_0) \) be the \( k \)th continuous approximation to the delta function at \( z_0 \) such that \( \delta_k(z - z_0) = 0 \) for all \( z \) in \( G \) with \( |z - z_0| > 1/k \). Then \( \delta_k(z - z_0) \) can be expressed by
\[
(\star) \quad \Psi_k(z) + \sum_{n=1}^{\infty} b^{(k)}_n \phi_n(z)
\]
for \( z \) in \( G \), where \( \Psi_k(z) \) is orthogonal to \( \phi_n(z) \) for \( n = 1, 2, \ldots \), on \( G \), has \( \int_G \left| \Psi_k(z) \right|^2 \, dA_z < \infty \) and is continuous on \( G \). By (4), the definition of \( a_{nm} \),
\[
\int_G \int_G \left[ V(z, \xi) - \sum_{n,m=1}^{\infty} a_{nm} \phi_n(z) \phi_m(\xi) \overline{[\Psi_k(z) + \sum_{n=1}^{\infty} b^{(k)}_n \phi_n(z)]} \right] \cdot \left[ \Psi_k(\xi) + \sum_{m=1}^{\infty} b^{(k)}_m \phi_m(\xi) \right] \, dA_z \, dA_t = 0.
\]
If \( z_0 \) is in \( G \), taking \( \lim_{k \to \infty} \) yields
\[
(5) \quad V(z_0, z_0) - \sum_{n,m=1}^{\infty} a_{nm} \phi_n(z_0) \phi_m(z_0) = 0.
\]
A similar computation using \( \delta_k(z - z_0) + \delta_k(z - z_1) \) for the test function yields
\[
V(z_0, z_0) - \sum_{n,m=1}^{\infty} a_{nm} \phi_n(z_0) \phi_m(z_0) + V(z_0, z_1) - \sum_{n,m=1}^{\infty} a_{nm} \phi_n(z_0) \phi_m(z_1) + V(z_1, z_0) - \sum_{n,m=1}^{\infty} a_{nm} \phi_n(z_1) \phi_m(z_0) + V(z_1, z_1) - \sum_{n,m=1}^{\infty} a_{nm} \phi_n(z_1) \phi_m(z_1) = 0.
\]
By (5) and the symmetry of $V$ and thus of $\alpha_{nm}$,

$$V(z, \xi) = \sum_{n, m=1}^{\infty} \alpha_{nm} \phi_n(z) \phi_m(\xi) \quad \text{for all } z \text{ and } \xi \text{ in } G.$$ 

Hence $V$ is analytic on $G \times G$.

It is now shown that the series (6) converge for $z$ and $\xi$ in $\mathcal{D}$.

Let $\theta_k(z) + \sum_{n=1}^{\infty} C_n \phi_n(z)$ be the representation of the $k$th continuous approximation to the delta function at $z_0$ where the representation holds for $z$ in $\mathcal{D}$, and $\theta_k(z)$ is orthogonal to $\phi_n(z)$ on $\mathcal{D}$ for $n = 1, 2, \ldots$.

\[
\int_{\mathcal{D}} \int_{\mathcal{D}} \left[ \sum_{n, m=1}^{L} \alpha_{nm} \phi_n(z) \phi_m(\xi) \right] \left[ \theta_k(z) + \sum_{n=1}^{\infty} C_n \phi_n(z) \right] \cdot \left[ \theta_k(\xi) + \sum_{m=1}^{\infty} C_m \phi_m(\xi) \right] dA_z dA_\xi \]

\[
\leq \int_{\mathcal{D}} \int_{\mathcal{D}} K_{\mathcal{D}}(z, \xi) \left[ \sum_{n=1}^{L} C_n \phi_n(z) \right] \left[ \sum_{m=1}^{L} C_m \phi_m(\xi) \right] dA_z dA_\xi \]

\[
\leq \int_{\mathcal{D}} \int_{\mathcal{D}} K_{\mathcal{D}}(z, \xi) \left[ \theta_k(z) + \sum_{n=1}^{\infty} C_n \phi_n(z) \right] \cdot \left[ \theta_k(\xi) + \sum_{m=1}^{\infty} C_m \phi_m(\xi) \right] dA_z dA_\xi \]
taking the $\lim_{k \to \infty}$

\[ K(\mathbf{z}_0, \mathbf{\tilde{z}}_0) \geq \left| \sum_{n,m=1}^{L} \alpha_{nm} \phi_n(\mathbf{z}_0) \phi_m(\mathbf{\tilde{z}}_0) \right| \quad \text{for all } L. \]

A similar computation using a representation of an approximation of the delta function at $\mathbf{z}_0$ plus the delta function at $\mathbf{z}_1$ and utilizing (7) yields

\[ K(\mathbf{z}_0, \mathbf{\tilde{z}}_0) + \text{Re} \ K(\mathbf{z}_0, \mathbf{\tilde{z}}_1) + K(\mathbf{z}_1, \mathbf{\tilde{z}}_1) \geq \left| \sum_{n,m=1}^{L} \alpha_{nm} \phi_n(\mathbf{z}_0) \phi_m(\mathbf{\tilde{z}}_1) \right|. \]

Hence $\left\{ \sum_{n,m=1}^{L} \alpha_{nm} \phi_n(\mathbf{z}) \phi_m(\mathbf{\tilde{z}}) \right\}_{L=1}^{\infty}$ is a normal family on $\mathbb{D} \times \mathbb{D}$. Since it converges to $V(\mathbf{z}, \mathbf{\tilde{z}})$ on $G \times G$, $\sum_{n,m=1}^{\infty} \alpha_{nm} \phi_n(\mathbf{z}) \phi_m(\mathbf{\tilde{z}})$ must converge to an analytic function on $\mathbb{D} \times \mathbb{D}$ that is a continuation of $V(\mathbf{z}, \mathbf{\tilde{z}})$.

**Corollary.** If $V(\mathbf{z}, \mathbf{\tilde{z}})$ is a symmetric, continuous, complex-valued function on $G \times G$, and

\[ \left| \sum_{n=1}^{L} \sum_{m=1}^{L} \alpha_n \alpha_m V(\mathbf{z}_n, \mathbf{z}_m) \right| \leq \sum_{n=1}^{L} \sum_{m=1}^{L} \alpha_n \alpha_m K(\mathbf{z}_n, \mathbf{\tilde{z}}_m) \]

for all complex vectors $(\alpha_1, \alpha_2, \cdots)$, and $(z_1, z_2, \cdots)$ with all $z_n$ in $G$, then $V(\mathbf{z}, \mathbf{\tilde{z}})$ is analytic in $G \times G$ and can be continued onto $\mathbb{D} \times \mathbb{D}$.

**References**


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