OSCILLATION OF SOLUTIONS OF CERTAIN ORDINARY DIFFERENTIAL EQUATIONS OF \( n \)TH ORDER

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Abstract. Necessary and sufficient conditions are given that all solutions of \( y^{(n)} + f(t, y) = 0 \) which are continuable to infinity are oscillatory in the case \( n \) is even and are oscillatory or strongly monotone in the case \( n \) is odd. The results generalize to arbitrary \( n \) recent results of J. Macki and J. S. W. Wong for the case \( n = 2 \) and include as special cases results of I. Kiguradze, I. Ličko and M. Švec, and Š. Belohorec.

The equation considered in this paper is

\[
y^{(n)} + f(t, y) = 0,
\]

where \( f(t, y) \) is defined in \( S = [0, \infty) \times (-\infty, \infty) \). Let \( F \) be the family of solutions of (1) which are indefinitely continuable to the right; i.e. if \( y(t) \in F \), then there exists \( t_0 \geq 0 \) such that \( y(t) \) exists on \( [t_0, \infty) \). A solution \( y(t) \) in \( F \) is said to be nonoscillatory if, for some \( T \) sufficiently large, \( y(t) \) is always positive or always negative for \( t \geq T \); otherwise a solution in \( F \) is oscillatory.

The first theorem generalizes to arbitrary \( n \geq 2 \) a theorem of Macki and Wong [6, Theorem 1] for the second order equation \( y'' + f(t, y) = 0 \), giving necessary and sufficient conditions for solutions of (1) in \( F \) to be oscillatory. This theorem also generalizes results of Kiguradze [2, Theorem 5] and Ličko and Švec [4] for the respective special cases \( y'' + yG(y^2, t) = 0 \), \( G(u, t) \) nonnegative and nondecreasing in \( u \), and \( y'' + a(t)y^\alpha = 0 \), \( \alpha > 1 \) and \( \alpha \) the ratio of odd integers. The second theorem generalizes results of Ličko and Švec [4] and Belohorec [1] for the latter equation when \( 0 \leq \alpha < 1 \). It also has points of contact with results of Kiguradze [3].

Assume for equation (1) that

(i) \( f(t, y) \) is continuous in \( S \);

(ii) \( a(t)\phi(y) \leq f(t, y) \) if \( y > 0 \) and \( f(t, y) \leq b(t)\psi(y) \) if \( y < 0 \), \( (t, y) \in S \),

where

(iii) \( a(t) \) and \( b(t) \) are nonnegative and locally integrable on \( [0, \infty) \) and neither \( a(t) \) nor \( b(t) \) is identically zero on any subinterval of \( [0, \infty) \).

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(iv) $\phi(y)$ and $\psi(y)$ are nondecreasing, and $y\phi(y) > 0$ and $y\psi(y) > 0$ on $(-\infty, \infty)$ for $y \neq 0$, and
(v) for some $\alpha \geq 0$,
\[
\int_a^\infty \frac{du}{\phi(u)} < \infty \quad \text{and} \quad \int_{-\alpha}^{-a} \frac{du}{\psi(u)} < \infty.
\]

Conditions (i) through (v) guarantee that equation (1) is strongly nonlinear [2].

**Theorem 1.** If the function $f(t, y)$ in (1) satisfies (i)-(v) and in addition
\[
\int_{-a}^\infty a(t) dt = \int_{-\alpha}^\infty b(t) dt = \infty,
\]
then if $n$ is even, each solution of (1) in $F$ is oscillatory, while if $n$ is odd, each solution in $F$ is either oscillatory or it tends monotonically to zero together with all its first $n-1$ derivatives.

For convenience, before proving Theorem 1 the possible behavior of a nonoscillatory solution is summarized in the following two lemmas [2, Lemma 1], [5, pp. 410, 418-419], the proofs of which are elementary.

**Lemma 1.** Suppose $f(t) \in C^k[a, \infty)$, $f(t) \geq 0$ and $f^{(k)}(t)$ is monotone. Then exactly one of the following is true:
(i) $\lim_{t \to -\infty} f^{(k)}(t) = 0$,
(ii) $\lim_{t \to -\infty} f^{(k)}(t) > 0$ and $f(t), \ldots, f^{(k-1)}(t)$ tend to $\infty$ as $t \to -\infty$.

**Lemma 2.** If $y(t) \in C^n[a, \infty)$, $y(t) \geq 0$ and $y^{(n)}(t) \leq 0$ on $[a, \infty)$, then exactly one of the following is true:
(I) $y'(t), \ldots, y^{(n-1)}(t)$ tend monotonically to zero as $t \to -\infty$,
(II) there is an odd integer $k$, $1 \leq k \leq n-1$, such that $\lim_{t \to -\infty} y^{(n-j)}(t) = 0$ for $1 \leq j \leq k-1$, $\lim_{t \to -\infty} y^{(n-k)}(t) \geq 0$, $\lim_{t \to -\infty} y^{(n-k-1)}(t) > 0$ and $y(t), y'(t), \ldots, y^{(n-k-2)}(t)$ tend to $\infty$ as $t \to -\infty$.

Analogous statements can be made if $y(t) \leq 0$ and $y^{(n)}(t) \geq 0$ on $[a, \infty)$.

**Proof of Theorem 1.** Suppose $y(t)$ is a nonoscillatory solution in $F$, say $y(t) > 0$ for $t \geq T \geq 0$. From (1),
\[
y^{(n)}(t) = -f(t, y(t)) \leq -a(t) \phi(y(t)).
\]
By Lemma 1, $y^{(n-1)}(t)$ decreases to a nonnegative limit, so from (3),
Suppose case I of Lemma 2 holds. Then an integration of \( n - 2 \) times from \( t \) to \( \infty \) yields

\[
(-1)^n y'(t) \geq \int_t^\infty \frac{(u - t)^{n-2}}{(n-2)!} a(u) \phi(y(u)) du.
\]

If \( n \) is even, integrating (5) from \( T \) to \( t \geq T \),

\[
y(t) \geq \int_T^t \frac{(u - T)^{n-1}}{(n-1)!} a(u) \phi(y(u)) du.
\]

Since \( \phi(u) \) is nondecreasing,

\[
\frac{\phi(y(t))}{\phi \left[ \int_T^t \frac{(u - T)^{n-1}}{(n-1)!} a(u) \phi(y(u)) du \right]} \geq 1,
\]

so, as in [6],

\[
\int_T^S \frac{(t - T)^{n-1}}{(n-1)!} a(t) dt,
\]

where

\[
R = \int_T^r \frac{(u - T)^{n-1}}{(n-1)!} a(u) \phi(y(u)) du
\]

and

\[
S = \int_T^r \frac{(u - T)^{n-1}}{(n-1)!} a(u) \phi(y(u)) du.
\]

If for some \( r \geq T \), \( R \geq \alpha \), then condition (v) gives a contradiction to condition (2), while if \( R < \alpha \) for all \( r \geq T \), then

\[
\alpha > R \geq \phi(y(T)) \int_T^r \frac{(u - T)^{n-1}}{(n-1)!} a(u) du,
\]

again in contradiction to condition (2).

If \( n \) is odd, then

\[
y'(t) \geq \int_t^\infty \frac{(u - t)^{n-2}}{(n-2)!} a(u) \phi(y(u)) du \geq 0,
\]

so \( y(t) \) decreases to a limit \( L \geq 0 \).
Suppose $L > 0$. Then integrating (6) from $T$ to $\infty$,

\[ y(T) > y(T) - L \geq \int_T^{\infty} \frac{(u - T)^{n-1}}{(n - 1)!} a(u)\phi(y)du \]

\[ \geq \phi(L) \int_T^{\infty} \frac{(u - T)^{n-1}}{(n - 1)!} a(u)du, \]

since $\phi(y)$ is nondecreasing in $y$. But this implies

\[ \int_0^{\infty} t^{n-1}a(t)dt < \infty. \]

Suppose now that case II of Lemma 2 holds. Proceeding as in case I,

(7) \[ y^{(n-k)}(t) \geq \int_t^{\infty} \frac{(u - t)^{k-1}}{(k - 1)!} a(u)\phi(y)du. \]

Since $y^{(j)}(t)$ increases to infinity, $j < n - k - 1$, there exists $t_1 \geq T$ such that $y^{(j)}(t) > 0$ for $t \geq t_1$, $j = 0, \ldots, n - k - 1$. Integrating (7) from $t_1$ to $t > t_1$,

\[ y^{(n-k+1)}(t) \geq \int_{t_1}^{t} \int_t^{\infty} \frac{(u - s)^{k-1}}{(k - 1)!} a(u)\phi(y(u))duds \]

\[ \geq \int_t^{\infty} \frac{(u - t_1)^k - (u - t)^k}{k!} a(u)\phi(y)du, \]

so

(8) \[ y^{(n-k+1)}(t) > \int_t^{\infty} \frac{(t - t_1)^k}{k!} a(u)\phi(y)du. \]

Integrating (8) from $t_1$ to $t$,

\[ y^{(n-k+2)}(t) > \int_t^{\infty} \frac{(t - t_1)^{k+1}}{(k + 1)!} a(u)\phi(y)du. \]

Proceeding in this fashion,

(9) \[ y'(t) > \int_t^{\infty} \frac{(t - t_1)^{n-2}}{(n - 2)!} a(u)\phi(y)du, \]

and a final integration from $t_1$ to $t$ gives

\[ y(t) > \int_{t_1}^{t} \frac{(u - t_1)^{n-1}}{(n - 1)!} a(u)\phi(y)du. \]
The proof now proceeds as in case I. Now suppose \( y(t) \) is a solution of (1) such that for \( t \geq T, y(t) < 0 \). The proof is the same as the case \( y(t) > 0 \) with \( a(t) \) and \( \phi(y) \) replaced respectively by \( b(t) \) and \( \psi(y) \) everywhere and with appropriate changes in the sense of inequalities.

Under the hypotheses of this theorem it is possible to have a non-oscillatory solution which tends monotonically to zero if \( n \) is odd and case I of Lemma 2 holds for this solution. For example, for \( n = 3 \) the equation

\[
y''' + e^t y^2 \text{sgn } y = 0
\]

has the solution \( y(t) = e^{-t} \). In this example one can choose \( \phi(y) = \psi(y) = y^2 \text{sgn } y, \alpha = 1 \) and \( a(t) = b(t) = e^t \).

Note. If \( \int_0^\infty t^{n-1} a(t) dt \) in (2) is finite and in condition (ii), \( a(t) \phi(y) \leq f(t, y) \leq b(t) \psi(y) \) simultaneously in \( S \), a solution in \( F \) which is non-oscillatory can be constructed exactly as in [6] making use of the integral equation

\[
y(t) = 1 + (-1)^{n-1} \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, y(s)) ds,
\]

and similarly, if \( \int_0^\infty t^{n-1} b(t) dt < \infty \), the integral equation

\[
y(t) = -1 + (-1)^{n-1} \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, y(s)) ds
\]

may be used to construct a nonoscillatory solution.

In the next theorem, condition (v) is changed so that equation (1) includes the special case

\[
y^{(n)} + a(t) y^\alpha = 0, \quad 0 \leq \alpha < 1,
\]

\( \alpha \) the ratio of odd integers.

Before stating the theorem the following lemma is given, a proof of which may be found in [3, Lemma 1].

**Lemma 3.** If \( y(t), y'(t), \ldots, y^{(n-1)}(t) \) are absolutely continuous and of constant sign on the interval \([t_0, \infty)\), and \( y^{(n)}(t)y(t) \leq 0 \), then there exists an integer \( l, 0 \leq l \leq n-1 \), which is even if \( n \) is odd and odd if \( n \) is even, so that

\[
|y(t)| \geq \frac{(t-t_0)^{n-1}}{(n-1) \cdots (n-l)} |y^{(n-1)}(2n-l-1)t|, \quad t \geq t_0.
\]
Theorem 2. Let \( f \) satisfy conditions (i)-(iv) and
(vi) there exist positive constants \( \lambda_0, M, N \) and constants \( \beta, \gamma, \) where
\( 0 \leq \beta < 1, \ 0 \leq \gamma < 1, \) such that
\[
\phi(\lambda y) \geq M\lambda^\beta \phi(y), \quad y > 0, \quad \lambda \geq \lambda_0 > 0.
\]
\[
\psi(\lambda y) \leq N\lambda^\gamma \psi(y), \quad y < 0.
\]
Then if
\[
\int \infty t^{(n-1)\beta} a(t) \, dt = \int \infty t^{(n-1)\gamma} b(t) \, dt = +\infty,
\]
each solution in \( F \) is oscillatory when \( n \) is even and each solution in \( F \)
is either oscillatory or tends to zero together with its first \( n-1 \) derivatives
if \( n \) is odd.

Proof. Suppose that \( n \) is even and there exists a nonoscillatory
solution \( y(t) \) such that \( y(t) > 0 \) for \( t \geq t_0 \). Then by Lemma 2, \( y'(t) \geq 0 \)
so \( y(t) \) is nondecreasing, and \( y^{(n)}(t) \leq 0 \) so \( y^{(n-1)}(t) \) is nonincreasing
and positive on \( [t_0, \infty) \). Therefore by Lemma 3,
\[
y(t) \geq y(2^{1-n} t) \geq A t^{n-1} y^{(n-1)}(t),
\]
\[
t \geq 2^n t_0 = t_1, \quad \text{where} \quad A = 2^{-n^2}/(n - 1)!.
\]
Because of condition (ii), \( y(t) \) must satisfy
\[
y^{(n)}(t) + a(t) \phi(y) \leq 0,
\]
and since \( y(t) \) is nondecreasing, \( ky(t) \geq \lambda_0 \) for \( k \geq \lambda_0 / y(t_1), \ t \geq t_1, \) and
\( \phi(y) \geq (ky)^\beta \phi(1/k) M \) by (vi).

Therefore, letting \( B = k^\beta \phi(1/k) M > 0, \) it follows that \( y^{(n)}(t) + B a(t) y^\beta \leq 0, \ t \geq t_1, \) and so from (11),
\[
y^{(n)}(t) + A^\beta B a(t)^{(n-1)\beta} [y^{(n-1)}(t)]^\beta \leq 0.
\]
Dividing by \( [y^{(n-1)}(t)]^\beta \) and integrating from \( t_1 \) to \( t, \)
\[
\int \frac{y^{(n-1)}(t_1)}{y^{(n-1)}(t)} \, dy + A^\beta B \int_{t_1}^t s^{(n-1)\beta} a(s) \, ds \leq 0.
\]
But, since
\[
0 > \int \frac{y^{(n-1)}(t)}{y^{(n-1)}(t_1)} \, dy \geq \int_0^c \frac{dy}{y^\beta}, \quad 0 < c < \infty,
\]
and the latter integral is finite for \( \beta < 1, \) this gives a contradiction of (13) as \( t \to \infty \) if \( \int \infty t^{(n-1)\beta} a(t) \, dt = +\infty. \) Thus \( y(t) \) must be oscillatory.
The case where $y(t) < 0$ for $t \geq t_0$ can be handled similarly and yields a contradiction to the fact that $\int t^{(n-1)\gamma} b(t) dt = + \infty$. The inequalities in (11) and (12) are reversed with $b(t)\phi(y)$ replacing $a(t)\phi(y)$, and the inequality in (13) is in the same direction but with $y$ replaced by $-y$.

If $n$ is odd and $y(t)$ does not approach zero, then $\left| y^{(n-1)}(t) \right|$ is still nonincreasing, so that

$$\left| y(t) \right| = \left| \frac{y(t)}{y(2^{1-n}t)} \right| \cdot \left| y(2^{1-n}t) \right| \geq \inf_{t \geq t_0} \left| \frac{y(t)}{y(2^{1-n}t)} \right| A \left| y^{(n-1)}(t) \right| t^{n-1}, \quad t \geq t_1,$$

hence $\left| y(t) \right| \geq B_1 t^{n-1} \left| y^{(n-1)}(t) \right|$ for constant $B_1$, and the preceding proof again yields a contradiction to the existence of a nonoscillatory solution in class $F$.

If conditions (ii) and (vi) are extended so that the inequalities there hold for all $y$, then by modifications of Kiguradze’s proofs [3, p. 773], [3, Lemma 5], it can be shown that all solutions of (1) are extendible to infinity under the conditions of Theorem 2, and if either

$$\int t^{(n-1)\beta} a(t) dt \quad \text{or} \quad \int t^{(n-1)\gamma} b(t) dt$$

is finite, a solution $y(t)$ of (1) can be exhibited such that $\lim_{t \to \infty} y^{(n-1)}(t) = C_0 \neq 0$. Hence, if (ii) and (vi) are valid for all $y$, condition (10) is necessary and sufficient for all solutions of (1) to oscillate if $n$ is even and for each solution either to oscillate or tend to zero if $n$ is odd.

References


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