SHORTER NOTES

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CURVATURE FORMS FOR 2-MANIFOLDS

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The Gauss Bonnet formula is a well-known necessary condition for a 2-form to be the curvature form of a Riemannian metric on a 2-manifold. It appears to be not so well known that it is also sufficient. Precisely,

THEOREM. *Let M be a compact, connected, orientable 2-dimensional manifold. Let $\bar{\Omega}$ be a 2-form on M. Then a necessary and sufficient condition for $\bar{\Omega}$ to be the curvature form of a Riemannian metric on M is that*

$$\int_M \bar{\Omega} = 2\pi\chi(M), \tag{1}$$

where $\chi(M)$ is the Euler characteristic of M.

PROOF. As we have remarked, the necessity of (1) is the Gauss Bonnet Theorem. For the sufficiency, let g be *any* Riemannian metric on M. We shall show that $\bar{\Omega}$ is in fact the curvature form of a Riemannian metric conformal to g (that is, of the form $e^{2\lambda}g$ for some C^∞ function λ on M). Let Ω be the curvature form for g. Then

$$\int_M (\Omega - \bar{\Omega}) = 0. \tag{2}$$

Now (2) is precisely the statement that $\Omega - \bar{\Omega}$ is orthogonal to the harmonic 2-forms on M. Thus by the Hodge Theorem, $\Omega - \bar{\Omega}$ is in the image of the Laplacian. Since $\Omega - \bar{\Omega}$ is a 2-form on a 2-manifold, this means that there is a 2-form β such that

$$\Omega - \bar{\Omega} = d * d * \beta.$$

Let $\lambda = *\beta$. That $\bar{\Omega}$ is then the curvature form of the metric $\bar{g} = e^{2\lambda}g$

Received by the editors September 26, 1969.

AMS Subject Classifications. Primary 5372, 5345; Secondary 5325, 5732.

Key Words and Phrases. Gauss-Bonnet, Riemannian manifold, curvature forms, conformal metric, Laplacian, Hodge Theorem.

[1] Supported in part by NSF grant GP 13850.

follows from the classical formula for change of curvature under conformal change of metric. We may compute this simply as follows. Let $\{\omega_1, \omega_2\}$ be a local oriented orthonormal coframe field on M for the metric g. If we set $\bar{\omega}_i = e^\lambda \omega_i$, then $\{\bar{\omega}_1, \bar{\omega}_2\}$ is a local oriented orthonormal coframe field for $\bar{g}$. Now $\Omega = d\phi_{12}$ where the Riemannian connection form ϕ_{12} is uniquely determined by the requirements that $\phi_{12} = -\phi_{21}$, $d\omega_1 = -\phi_{12} \wedge \omega_2$, and $d\omega_2 = -\phi_{21} \wedge \omega_1$. We compute $\bar{\phi}_{12}$. Let $d\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$. Then

$$d\bar{\omega}_1 = e^\lambda(d\lambda \wedge \omega_1 - \phi_{12} \wedge \omega_2) = -(\lambda_2\omega_1 - \lambda_1\omega_2 + \phi_{12}) \wedge \bar{\omega}_2.$$

Thus $\bar{\phi}_{12} = \lambda_2\omega_1 - \lambda_1\omega_2 + \phi_{12} = \phi_{12} - *d\lambda$. Thus the curvature of the metric $\bar{g} = e^{2\lambda}g$ is $d\bar{\phi}_{12} = d\phi_{12} - d*d\lambda = \Omega - d*d\lambda = \bar{\Omega}$.

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