EXTENSION OF A RESULT OF DIEUDONNÉ[1]

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ABSTRACT. Dieudonné showed that there exists a normal (countably) compact uniform T_1-space which has no topology preserving complete uniformity [4]. His example, being the space of the countable ordinals with respect to the order topology, everywhere locally has a complete uniformity. Here we show, as a *corollary* to Dieudonné's result and a result of Worrell [10], that there exists a normal (countably) compact first countable involutorily homogeneous uniform T_1-space locally homeomorphic with itself which has no topology preserving complete uniformity.

1. Definitions and notation. Terminology not defined here generally is much as in [6], *space* being used equivalently with *topological space*. As in [6] if $\prec$ is a relation, $x \prec y$ means that the two term sequence (x, y) belongs to $\prec$. If A and B are sets, $A \cdot B$ denotes their intersection and $A+B$ denotes their sum or union. As in [7] if K is a collection of sets, K^* denotes the sum (union) of the sets of K. In saying that a space S is *involutarily homogeneous* it is meant that if X and Y are distinct points of S, there exists a homeomorphism θ of S onto itself taking X into Y such that $\theta[\theta(P)] = P$ for each point P of S [3]. A space S is said to be *locally homeomorphic* with itself provided that if P is a point of an open set D of S, there exists a homeomorphism of S onto an open subset of D containing P. For *collectionwise normal space*, see [2]. *Compact space* is taken in the sense of [5] and not in the sense of [6]. In the context of T_1-spaces, those spaces called *compact* herein would be called *countably compact* in [6]. Spaces called *compact* in [6] are herein called *bicompact* following the usage of Alexandroff-Urysohn. For *base of countable order*, see [1], [9]. The following definition of arc is motivated by that of R. L. Moore [7, p. 39]. An *arc* is a nondegenerate bicompact connected T_1-space having no more than two noncut points. It follows that such a space is Hausdorff and has exactly two noncut points. It can be proved that an arc in this sense is a homeomorph of the interval $[0, 1]$ of the real numbers taken in the usual topology if and only if it has a base

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of countable order. The terminology $\aleph$-*wise Lindelöfian* is taken in a natural sense.

2. Derivation.

THEOREM. *For each $\aleph > \aleph_0$ there exists a compact normal uniform T_1-space S locally homeomorphic with itself satisfying these conditions:*

(1) *S has a topology preserving complete uniformity nowhere locally.*

(2) *If X and Y are points of S, there exists a homeomorphism θ of S onto itself taking X into Y and leaving no point fixed such that $\theta[\theta(P)] = P$ for each point P of S.*

(3) *S has a base of countable order.*

(4) *$\aleph$ is the least cardinal number $\aleph'$ such that S is $\aleph'$-wise Lindelöfian.*

PROOF. Let Γ denote a set of cardinal number $\aleph > \aleph_0$. Let $\prec_\Gamma$ denote a subset of $\Gamma \times \Gamma$ with respect to which Γ is well ordered in such a way that it is covered by an uncountable collection Σ of mutually exclusive segments satisfying these conditions:

(1) If σ belongs to Σ, $\bar{\bar{\sigma}} = \aleph$; and if σ' is a proper initial segment of σ, then $\bar{\bar{\sigma}}' < \aleph$.

(2) If $\prec_\Sigma$ is the subset of $\Sigma \times \Sigma$ such that $\sigma \prec_\Sigma \sigma'$ if and only if σ and σ' are members of Σ such that the elements of σ precede the elements of σ', then all proper initial segments of Σ with respect to $\prec_\Sigma$ are countable.

Let ψ denote the order topology for Γ. Let Γ' denote the set of all elements of Γ at which (Γ, ψ) is first countable. Let ψ' denote the topology for Γ' induced by ψ. Let A_1 denote the set of all elements of Γ' that are limit points of Γ' with respect to ψ'.

There exists a sequence $f_1, f_2, \cdots$ of reversible transformations satisfying these conditions:

(1) For each n, the range R_n of f_n is a family of mutually exclusive sets W not intersecting Γ' well ordered with respect to a subset $\prec_W$ of $W \times W$ and constituting the domain of a reversible transformation ϕ_W having Γ' as its range such that if $x \prec_W y$ then $\phi_W(x) \prec_\Gamma \phi_W(y)$.

(2) $\Gamma' - A_1$ is the domain D_1 of f_1. If $n > 1$ and A_n denotes the sum of the sets $\phi_W^{-1}(A_1)$ for all sets W belonging to R_{n-1}, then the domain D_n of f_n is $R_{n-1}^* - A_n$.

(3) If $n < k$, R_n^* does not intersect R_n^*.

For each α in $A_1 + A_2 + \cdots$, let q_α denote the n such that A_n contains α. Let $P_{1,\alpha}, P_{2,\alpha}, \cdots$ denote the sequence such that

(1) if $n < q_\alpha$, $P_{n,\alpha}$ belongs to D_n,

(2) if $n < q_\alpha$, $P_{n+1,\alpha}$ belongs to $f_n(P_{n,\alpha})$, and

(3) if $n \geqq q_\alpha$, $P_{n,\alpha}$ is α.

Let S denote the set of all sequences $x_1, x_2, \cdots$ satisfying one of these conditions:

(1) Each x_n belongs to D_n and each x_{n+1} belongs to $f_n(x_n)$.

(2) For some α in $A_1+A_2+\cdots$, each x_n is $P_{n,\alpha}$.

Let $\prec_1$ denote $\prec_\Gamma\cdot(\Gamma'\times\Gamma')$. If $n\geqq 1$, let $\prec_{n+1}$ denote the subset of $R_n\times R_n$ such that if x_{n+1} and y_{n+1} belong to R_n then $x_{n+1}\prec_{n+1} y_{n+1}$ if and only if it is true that whenever $x_1, \cdots, x_{n+1}$ and $y_1, \cdots, y_{n+1}$ are sequences such that for all $i\leqq n$ (1) x_i and y_i belong to D_i and (2) x_{i+1} belongs to $f_i(x_i)$ and y_{i+1} belongs to $f_i(y_i)$ then

(1) for some $i\leqq n+1$, $x_i\neq y_i$,

(2) if $x_1\neq y_1$, $x_1\prec_1 y_1$, and

(3) if the least $i\leqq n+1$ such that $x_i\neq y_i$ exceeds 1, $x_i\prec_{f_{i-1}(x_{i-1})} y_i$.

For each n, D_n+A_n is well ordered with respect to $\prec_n$. Let $\prec_S$ denote the subset of $S\times S$ such that if $b_1, b_2, \cdots$ and $c_1, c_2, \cdots$ belong to S, then $b_1, b_2, \cdots \prec_S c_1, c_2, \cdots$ if and only if

(1) there exists some n such that $b_n\neq c_n$ and

(2) if k is the least n such that $b_n\neq c_n$, $b_k\prec_k c_k$.

The transitive asymmetric relation $\prec_S$ connects S. Moreover, for each α in $A_1+A_2+\cdots$, there exists a sequence $M_{1,\alpha}, M_{2,\alpha}, \cdots$ of nonrepeating sequences belonging to S such that

(1) for each n, $M_{n,\alpha}\prec_S M_{n+1,\alpha}\prec_S P_{1,\alpha}, P_{2,\alpha}, \cdots$,

(2) if $u\prec_S P_{1,\alpha}, P_{2,\alpha}, \cdots$ there exists some n such that $u\prec_S M_{n,\alpha}$,

(3) if $q_\alpha>1$, the q_αth terms of the sequences $M_{n,\alpha}$ belong to $f_{q_{\alpha-1}}(P_{q_{\alpha-1},\alpha})$, and

(4) for each n, if $i\geqq q_\alpha$, d is the ith term of $M_{n,\alpha}$ and d' is the $i+1$th term of $M_{n,\alpha}$, then with respect to $\prec_{i+1}$, d' is the first element of $f_i(d)$.

If $k\geqq 1$, by *a region of type* I *of index k* is meant a set R such that for some d in D_k, R is the set of all sequences in S having d as kth term. By a *region of type* II *of index k* is meant a set R such that for some α in $A_1+A_2+\cdots$, R is the set of all sequences u in S such that $M_{k,\alpha}\preceq_S u \preceq_S P_{1,\alpha}, P_{2,\alpha}, \cdots$. Let τ denote the collection of all sets which are the sum (union) of some sets R such that for some n, either R is a region of type I of index n or R is a region of type II of index n. The topological space (S, τ), *hereafter denoted by* S, is shown in [10] to be a compact hereditarily collectionwise normal space locally homeomorphic with itself satisfying conditions (2), (3), and (4). Additionally the following conditions are fulfilled:

(a) If P is a point of an open set D of S, there exists a subset M of D containing P which in its relative topology is homeomorphic with the space Ω of countable ordinals with the order topology.

(b) S is a subspace of an arc.

From condition (b) or the complete regularity of S it may be seen that S is a uniform T_1-space. From condition (3) it follows that S is first countable. With application, additionally, of the facts that S is T_2 and Ω is compact, it may be seen that any M as in (a) is closed with respect to S. Closed subspaces of complete uniform spaces are complete [6]; hence if S is a complete uniform space, so is M in the relative topology. But M is homeomorphic with Ω and thus, by Dieudonné's cited result, has no topology preserving complete uniformity. Thus S is not a complete uniform space. Similarly S is nowhere locally a complete uniform space.

REMARK. Since (1) S is a regular T_0-space having a base of countable order and (2) S has a base such that the closures of the elements of any monotonic subcollection of B have a common part, S is an open continuous image of some metrically topologically complete space [8]. It is interesting to note, in this connection, the Corollary of [6, p. 203]. We see here, moreover, a striking example of an open continuous mapping of a Čech complete space onto a non Čech complete normal T_1-space. For S, as constructed above, is nowhere locally Čech complete [10].

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