A PROOF OF WHITMAN'S REPRESENTATION THEOREM FOR FINITE LATTICES[1]

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ABSTRACT. The theorem to be proved states that every finite lattice is isomorphic to a sublattice of the lattice $\mathcal{E}(S)$ of all equivalence relations on a countable set S. Our proof combines concreteness with freedom from long routine computations.

Earlier proofs appear in [1] and [2]. Let L be a finite lattice. Without loss of generality assume that the elements of L are subsets of $\{1, \cdots, m\}$, that the unit and zero of L are $\{1, \cdots, m\}$ and ϕ respectively, and that the meet operation of L is set intersection. Write $\vee$ for the join in L, $\cup$ for set union. If $r=(a_1, \cdots, a_m)\in N^m$ define $(r)_l=a_l$ $(l=1, \cdots, m)$; if r, $s\in N^m$ and $\alpha\in L$ define $r\equiv_\alpha s \Leftrightarrow (r)_l=(s)_l$ for all $l\in\alpha$; finally let $\gamma(r, s)$ be the greatest $\gamma\in L$ such that $r\equiv_\gamma s$ if there is a greatest such γ, $\gamma(r, s)$ undefined otherwise.

It is sufficient to find a set $S\subseteq N^m$ such that the equivalence relations $\equiv_\alpha$ form a sublattice of $\mathcal{E}(S)$ *dually* isomorphic to L, which will certainly be the case if:

I. For each α, $\beta\in L$ with $\alpha\nleq\beta$ there are r, $s\in S$ such that $r\equiv_\beta s$ but $r\not\equiv_\alpha s$.

II. $\gamma(r, s)$ is defined for all r, $s\in S$.

III. For each $r, s\in S$ and $\alpha, \beta\in L$ such that $r\equiv_{\alpha\cap\beta}s$, there are u, v, $w\in S$ such that $r\equiv_\alpha u\equiv_\beta v\equiv_\alpha w\equiv_\beta s$.

Now it is easy to construct a finite $S_0\subseteq N^m$ satisfying I and II. Suppose $S_n\subseteq N^m$ is finite and satisfies I and II. If S_n also satisfies III let $S_{n+1}=S_n$. If not let r, $s\in S_n$ and α, $\beta\in L$ be the least (in some previously fixed well-ordering of type ω of $N^m\times N^m\times L\times L$) counterexample to III. Define $w\in N^m$ by

$$\begin{aligned}(w)_l &= (s)_l \qquad \text{if } l\in\beta,\\ &= b \qquad\ \ \text{if } l\notin\beta\end{aligned}$$

where $b\in N$ is distinct from all $(q)_l$ with $q\in S_n$. Then for all $q\in S_n$, $\gamma(q, w)=\beta\cap\gamma(q, s)$. Thus $S_n'=S_n\cup\{w\}$ still satisfies I and II. Now define u, $v\in N^m$ by

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$$\begin{aligned} (u)_l &= (r)_l && \text{if } l \in \alpha, & (v)_l &= (w)_l && \text{if } l \in \alpha, \\ &= c && \text{if } l \in \beta - \alpha, & &= c && \text{if } l \in \beta - \alpha, \\ &= d && \text{if } l \notin \alpha \cup \beta, & &= e && \text{if } l \notin \alpha \cup \beta, \end{aligned}$$

where c, d, e are distinct natural numbers not equal to any $(q)_l$ for $q \in S_n'$. Then $\gamma(u, v) = \beta$ (since if $u \equiv_\gamma v$ then $\gamma \subseteq \alpha \cup \beta$ and $\gamma \cap \alpha \subseteq \gamma(r, w) = \alpha \cap \beta$ so by set algebra $\gamma \subseteq \beta$), and for all $q \in S_n'$, $\gamma(q, u) = \alpha \cap \gamma(q, r)$ and $\gamma(q, v) = \alpha \cap \gamma(q, w)$. Let $S_{n+1} = S_n' \cup \{u, v\}$. Then S_{n+1} is finite and satisfies I and II, moreover $r \equiv_\alpha u \equiv_\beta v \equiv_\alpha w \equiv_\beta s$.

Plainly $S = \bigcup_{n \geqq 0} S_n$ satisfies I, II, and III. We note that S is recursive provided that reasonable choices are made for the well-ordering and for b, c, d, and e; and that the representation is of type 3 in the sense of [1].

References

1. Bjarni Jónsson, *On the representation of lattices*, Math. Scand. **1** (1953), 193–206. MR **15**, 389.

2. Philip M. Whitman, *Lattices, equivalence relations, and subgroups*, Bull. Amer. Math. Soc. **52** (1946), 507–522. MR **8**, 62.

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