SOME ASYMPTOTIC THEOREMS FOR ABSTRACT DIFFERENTIAL EQUATIONS

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ABSTRACT. We consider in this paper results on differential equations with time independent operators; uniqueness of solutions which are bounded in the Stepanoff norm as well as weak almost-periodic solutions are some of the topics here considered.

Introduction. In this paper, which is closely related with some of our previous publications, a number of results concerning differential equations in Hilbert and Banach spaces are derived. They concern asymptotic behaviour, boundedness and almost-periodicity.

1. Our first result, a very simple one, is "essentially" the Theorem 1 in [1]. Here it is given in its natural, operator case framework.

THEOREM 1. *Let H be a Hilbert space, A a closed linear operator in H with dense domain $D(A)$; A^* be its adjoint operator, and suppose that for a real β the relations*

$$\text{(1.1)} \qquad \begin{aligned} \operatorname{Re}(Ax, x) &\leqq \beta(x, x), \qquad \forall x \in D(A), \\ \operatorname{Re}(A^*y, y) &\leqq \beta(y, y), \qquad \forall y \in D(A^*) \end{aligned}$$

are verified. Let $u(t)$, $t \geqq 0 \to D(A)$, be a strong solution of equation

$$\text{(1.2)} \qquad u'(t) = Au(t).$$

Then $\|u(t)\| \leqq e^{\beta t}\|u(0)\|$ holds, $\forall t \geqq 0$.

PROOF. Let us put $v(t) = e^{-\beta t}u(t)$. Then $v'(t) = (A - \beta I)v(t)$. It is easy to see that $A - \beta I$ is the infinitesimal generator of a strongly continuous one-parameter semigroup $T_\beta(t)$ such that $\|T_\beta(t)\| \leqq 1$. We see also the representation $v(t) = T_\beta(t)v(0)$; consequently $\|v(t)\| \leqq \|v(0)\| = \|u(0)\|$ and $\|u(t)\| \leqq e^{\beta t}\|u(0)\|$.

Now, a very simple result about asymptotic behaviour (cf. Theorem 2 in [1]) is the

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THEOREM 2. *Let us have* (1.1) *with* $\beta<0$. *Then if* $f_0\in H$ *is given and* $u(t)$, $t\geqq 0\to D(A)$ *is solution of*

$$(1.3) \qquad u'(t) = Au(t) + f_0.$$

There exists $w_0\in H$ *such that* $\lim_{t\to\infty} u(t)=w_0$.

PROOF. It follows easily that A^{-1} exists and belongs to $\mathfrak{L}(H, H)$. We consider $w_0=-A^{-1}f_0$, and put $v(t)=u(t)-w_0$. We have $v'(t)=Av(t)$; apply Theorem 1 and we get, as $\beta<0$, $\lim_{t\to\infty} u(t)=w_0$.

Our next result is a simple generalization of Lemma 1 in our paper [2].

THEOREM 3. *Let* $\mathfrak{X}$ *be a Banach space, and* $T_t\in\mathfrak{L}(\mathfrak{X},\mathfrak{X})$, $t\geqq 0$, *be a one parameter strongly continuous semigroup, such that* $\|T_t\|\leqq Me^{\beta t}$, $\beta<0$, $t\geqq 0$. *Let* A *be its infinitesimal generator and* $u(t)$, $-\infty<t<+\infty\to D(A)$, *be a strong solution of* $u'(t)=Au(t)$. *Then if* $\sup_{t\in R^1}\int_t^{t+1}\|u(\sigma)\|^2 d\sigma<\infty$ *it follows* $u(t)=\theta$ *for every real* t.

REMARK. Similar results are given in our paper [3].

PROOF. We see firstly the representation: $u(t)=T_{t-t_0}u(t_0)$, $\forall t\geqq t_0$. Then we remark existence of a sequence $(t_n)_1^\infty$ such that $\lim_{n\to\infty} t_n=-\infty$ and such that $\sup_{n\in\mathfrak{N}}\|u(t_n)\|=L<\infty$. Next for arbitrary real t we take n large enough in order to have $t_n<t$ and consequently $u(t)=T_{t-t_n}u(t_n)$. So we derive $\|u(t)\|\leqq Me^{\beta(t-t_n)}\cdot L$; for $n\to\infty$ we get $u(t)=0$.

2. In this section we give a complement to our result on almost-periodicity of certain relatively-compact valued vector functions (see [4]) by taking into account weakly almost-periodic solutions (of abstract differential equations). Remember that if $\mathfrak{X}$ is a Banach space and $\mathfrak{X}^*$ its strong dual, a continuous function $f(t)$, $-\infty<t<+\infty\to\mathfrak{X}$ is weakly almost-periodic when $\langle x^*, f(t)\rangle$ is Bohr-almost-periodic for every $x^*\in\mathfrak{X}^*$. Our result is the following

THEOREM 4. *In the Banach space* $\mathfrak{X}$, *consider a strongly continuous one-parameter semigroup* $T_t\in\mathfrak{L}(\mathfrak{X},\mathfrak{X})$ *such that* $\lim_{t\to\infty} T_t x=\theta$, $\forall x\in\mathfrak{X}$. *Let also* $Q\in\mathfrak{L}(\mathfrak{X},\mathfrak{X})$ *be a compact operator commuting with* T_t, $\forall t\geqq 0$. *Its inverse* Q^{-1} *exists on a dense set in* $\mathfrak{X}$, *and the adjoint* $(Q^{-1})^*$ *is defined on a dense set in* $\mathfrak{X}^*$. *Let* A *be the infinitesimal generator of* T_t; $f(t)$ *a continuous weakly almost-periodic function* $-\infty<t<+\infty\to\mathfrak{X}$; $u(t)$ *a strong solution, on the whole real axis of equation* $u'(t)=Au(t)+f(t)$, *such that* $\sup_{t\in R^1}\|u(t)\|<\infty$. *Then* $u(t)$ *is weakly almost-periodic.*

PROOF. We remark first, as a standard result, the representation formula

$$u(t) = T_{t-t_0}u(t_0) + \int_{t_0}^{t} T_{t-s}f(s)ds, \qquad t \geqq t_0.$$

Next we see that: if $g(t)$, $-\infty < t < +\infty \to \mathfrak{X}$, is a bounded function such that $\langle x^*, g(t)\rangle$ is almost-periodic for a dense set of elements in the dual space $\mathfrak{X}^*$, then $g(t)$ is weakly almost-periodic. The result is a corollary of the fact that uniform convergent on R^1 sequences of almost-periodic functions have almost periodic limit. A simple remark now is that $w(t) = Qu(t)$ has representation

$$w(t) = T_{t-t_0}w(t_0) + \int_{t_0}^{t} T_{t-s}(Qf)(s)ds, \qquad t \geqq t_0$$

and that range of $w(t)$ is relatively compact in $\mathfrak{X}$ as $t \in R^1$. Then we have

LEMMA. *If $h(t)$ is continuous weakly almost-periodic, $t \in R^1 \to \mathfrak{X}$, and if Q is a compact operator $\in \mathfrak{L}(\mathfrak{X}, \mathfrak{X})$ then Qh is strongly almost-periodic.*

In fact $h(t)$ is bounded, hence Qh has relatively compact range. Moreover $Qh(t)$ is weakly almost-periodic too; by well-known facts $Qh(t)$ is strongly almost-periodic.

By this Lemma, Qf is almost-periodic. We apply our Theorem 1 in [4] and obtain that $w(t) = Qu(t)$ is almost-periodic. Then $u(t) = Q^{-1}w(t) = Q^{-1}Qu(t)$. We take now $x^* \in D((Q^{-1})^*)$ (which is dense in $\mathfrak{X}^*$).

We have $\langle x^*, Q^{-1}Qu(t)\rangle = \langle (Q^{-1})^*x^*, w(t)\rangle$ which is almost-periodic Bohr. From the above made remarks, $u(t)$ is weakly almost-periodic.

3. Here we remember a certain natural generalization of almost-periodic functions.

DEFINITION 3.1. Let $h(t)$, $0 \leqq t < \infty$, be a continuous function with values in the Banach space $\mathfrak{X}$. We say that $h(t)$ is in the class $\mathfrak{B}_{\mathfrak{X}}^+$ when the set of translates $(h(t+\eta))_{\eta \geqq 0}$ is a relatively compact set in the space $C[0, \infty; \mathfrak{X}]$.

DEFINITION 3.2. Let $h(t)$ be a continuous function, $0 \leqq t < \infty \to \mathfrak{X}$. We say that $h(t)$ is in the class $\mathfrak{F}_{\mathfrak{X}}^+$ when $\forall \epsilon > 0$, $\exists L_\epsilon > 0$, $N_\epsilon > 0$, such that in every interval $[a, a+L] \subset [0, \infty)$, $\exists \zeta_\epsilon$ with property

$$\sup_{t > N_\epsilon} \|h(t + \zeta_\epsilon) - h(t)\|_{\mathfrak{X}} < \epsilon.$$

In the Appendix of our paper [1] a proof of the inclusion $\mathfrak{B}_{\mathfrak{X}}^+ \subset \mathfrak{F}_{\mathfrak{X}}^+$ is indicated.

DEFINITION 3.3. A continuous function, $0 \leqq t < \infty \to \mathfrak{X}$, $h(t)$ is called weakly-$\mathfrak{B}_{\mathfrak{X}}^+$ (resp. weakly -$\mathfrak{F}_{\mathfrak{X}}^+$) if, for each $x^* \in \mathfrak{X}^*$, $\langle x^*, h(t)\rangle$ is in class $\mathcal{B}^+$ (resp $\mathfrak{F}^+$) corresponding to $\mathfrak{X}$ = scalar field. It is easy to see that if $g(t) \in \mathfrak{F}^+_{\mathfrak{X}}$, then, $\forall x^* \in \mathfrak{X}^*$, $\langle x^*, g(t) \rangle \in \mathfrak{F}^+$. Also, we have the standard proof of the fact that uniform limits on $0 \leqq t < \infty$ of sequences $(h_n(t))_1^\infty \subset \mathfrak{F}^+_{\mathfrak{X}}$ belong to the same class. We do now a simple observation, connected with Theorem 5 in [1]. We have precisely the

THEOREM 5. *Let $\mathfrak{X}$ be a Banach space; T_t, $t \geqq 0 \to \mathfrak{L}(\mathfrak{X}, \mathfrak{X})$, be a strongly continuous one-parameter semigroup, such that $\|T_t\| \leqq M$, $t \geqq 0$. Let A be its infinitesimal generator and $u(t)$, $t \geqq 0 \to D(A)$, be a strong solution of the equation: $u'(t) = Au(t)$, $t \geqq 0$. Suppose that $u(t)$ has relatively compact trajectory; then $u(t) \in \mathfrak{F}^+_{\mathfrak{X}}$.*

PROOF. We have as usual, representation $u(t) = T_t u(0)$, $t \geqq 0$. We prove that $u(t) \in \mathcal{B}^+_{\mathfrak{X}}$. Consider the set of vector-functions: $\{u(t+\eta)\}_{\eta \geqq 0} = \{T_{t+\eta} u(0)\}_{\eta \geqq 0}$. By relative compactness we may find a sequence $(\eta_n)_1^\infty \subset [0, \infty)$ such that $(T_{\eta_n} u(0))_{n=1}^\infty$ is a Cauchy sequence in $\mathfrak{X}$. Then $\{u(t+\eta_n)\}_{n=1}^\infty$ is a Cauchy sequence in $C[0, \infty; \mathfrak{X}]$. This follows from the obvious estimate:

$$\begin{aligned} \|T_{t+\eta_n} u(0) - T_{t+\eta_m} u(0)\| &\leqq \|T_t\| \, \|T_{\eta_n} u(0) - T_{\eta_m} u(0)\| \\ &\leqq M \|u(\eta_n) - u(\eta_m)\|. \end{aligned}$$

We complement this result by another one, on weak-$\mathfrak{F}^+_{\mathfrak{X}}$ solutions.

THEOREM 6. *Let $\mathfrak{X}$ be a Banach space; T_t, $t \geqq 0 \to \mathfrak{L}(\mathfrak{X}, \mathfrak{X})$, be a strongly continuous one-parameter semigroup such that $\|T_t\| \leqq M$, $t \geqq 0$. Let A be its infinitesimal generator; suppose that for a complex λ_0, operator $(\lambda_0 - A)^{-1}$ is a linear compact operator in $\mathfrak{X}$; suppose also the adjoint operator A^* be densely defined in $\mathfrak{X}^*$. Consider then $u(t)$, $t \geqq 0 \to D(A)$ a strong solution of $u'(t) = Au(t)$, $t \geqq 0$, such that $\|u(t)\| \leqq M$, $t \geqq 0$. Then $u(t)$ is weakly-$\mathfrak{F}^+_{\mathfrak{X}}$.*

PROOF. We have again: $u(t) = T_t u(0)$, $t \geqq 0$. Denote by $v(t)$ the vector-function $(\lambda_0 - A)^{-1} u(t)$. Because T_t commutes with $(\lambda_0 - A)^{-1}$ we obtain $v(t) = T_t v(0)$, and moreover $v(t)$ has relatively compact trajectory. We apply the previous theorem and get $v(t) \in \mathfrak{F}^+_{\mathfrak{X}}$. Hence $u(t) = (\lambda_0 - A) v(t)$. Take then $x^* \in D(A^*)$. We have $\langle x^*, u(t) \rangle = \langle x^*, (\lambda_0 - A) v(t) \rangle = \langle (\lambda_0 - A)^* x^*, v(t) \rangle = \langle y^*, v(t) \rangle$. Applying the previous remarks our result follows.

We end this paper giving, in a concrete case an effective criterium in order that for a given semigroup T_t, the trajectory $\{T_t x\}_{t \geqq 0}$ be relatively compact (see Theorem 5). Consider the space $L^p(R^p)$, $1 \leqq p < \infty$. Remember a necessary and sufficient condition for a set $\mathfrak{A} \subset L^p(R^n)$ to be relatively compact:

(i) $\int_{R^n} |u(x)|^p dx \leqq M$, $\forall u \in \mathfrak{A}$,

(ii) $\lim_{\rho\to\infty} \int_{R^n} |u_\rho(x)|^p\, dx = 0$ uniformly on $u \in \mathfrak{A}$; here $u_\rho(x) = u(x)$, $|x| > \rho$ and $u_\rho(x) = 0$ for $|x| \leqq \rho$.

Call t_ρ; $\phi \to \phi_\rho$, $L^p \to L^p$, the truncation operator,

(iii) $\lim_{|h|\to 0} \int_{R^n} |(\zeta_h u - u)(x)|^p\, dx = 0$, uniformly on $u \in \mathfrak{A}$. Here $(\zeta_h u)(x) = u(x+h)$ is the translation operator. Then we have

THEOREM 7. *Let T_t, $t \geqq 0 \to \mathfrak{L}(L^p(R^n), L^p(R^n))$, $1 \leqq p < \infty$, be a strongly continuous semigroup such that $\|T_t\| \leqq M$, $t \geqq 0$. Suppose that T_t commutes with the truncation operator t_p, for each $\rho > 0$ and with the translation operator ζ_h for each $h \in R^n$. Then the set $\mathfrak{A} = \{T_t\phi_0\}_{t\geqq 0}$ is, for fixed $\phi_0 \in L^p(R^n)$, a relatively compact set in $L^p(R^n)$.*

The proof is immediate if we apply the previous criterium.

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