

EXTENDING HOMEOMORPHISMS IN COMPACTIFICATION OF FRÉCHET SPACES

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ABSTRACT. In the study of extending homeomorphisms in the compactification of Fréchet spaces into the Hilbert cube Q , it is shown that for a given homeomorphism f of s onto itself and a closed subset K of s satisfying property Z in s , there is a homeomorphism g of s onto itself such that $gfg^{-1}|_{\sigma(K)}$ extends to a homeomorphism of Q onto Q . The proof is by employing the rather useful shifting homeomorphism on Q .

1. Let F be a separable infinite-dimensional Fréchet space. Let Q denote the Hilbert cube $\prod_{i=1}^{\infty} J_i$, $J_i = [-1, 1]$, and let $s = \prod_{i=1}^{\infty} \text{Int } J_i$. s is sometimes called the pseudo-interior of Q . It has been shown that there is a compactification μ of F into Q such that $\mu(F) = s$, [4], [5] and [6].

Let N denote the set of integers and for each $i \in N$, let Q_i denote a copy of Q . We now identify Q as $\prod_{i \in N} Q_i$ and s as $\prod_{i \in N} s_i$ where s_i is the pseudo-interior of Q_i . Let p_i denote the point $(0, 0, \dots)$ of Q_i or s_i and let $p = (0, 0, \dots)$ of Q or s .

The *shifting* homeomorphism Φ of Q into itself is defined by

$$\Phi(\dots, x_{-1}, x_0, x_1, \dots) = (\dots, x'_{-1}, x'_0, x'_1, \dots)$$

where $x_i, x'_i \in Q_i$ and $x'_i = x_{i+1}$, for all $i \in N$.

Φ may be regarded as a *universal β^* -homeomorphism* on Q in the sense of the following main theorem.

THEOREM. *For any closed set A in F and any imbedding f of A into F such that $f(A)$ is closed and both A and $f(A)$ have property Z in F , there is an imbedding μ of F into Q such that $\mu(F) = s$ and $\mu f \mu^{-1}|_{\mu(A)} = \Phi|_{\mu(A)}$.*

Following Anderson [2], a closed set K in a space X has *property Z* in X if for each homotopically trivial nonnull open subset U of X , $U \setminus K$ is both nonnull and homotopically trivial. A homeomorphism h of Q onto Q is a *β^* -homeomorphism* if h carries s onto s .

COROLLARY 1. *Let A, B be closed sets with property Z in s and let f be a homeomorphism of A onto B . Then there is a homeomorphism g of s*

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onto itself such that $gfg^{-1}:g(A)\rightarrow g(B)$ extends to a β^* -homeomorphism of Q onto Q .

COROLLARY 2. Let A, B be closed sets with property Z in s . Then there is a homeomorphism h of s onto s such that $cl(h(A)), cl(h(B))$ are homeomorphic (closures are taken with respect to Q).

2. Before we prove the theorem we need the following lemma. Let $X = s_1 \times s_2$ where $s_1 = s = s_2$. For $i = 1, 2$, let p_i denote the point $(0, 0, \dots)$ of s_i and let π_1 be the projection function of X onto $s_1 \times p_2$.

LEMMA. Suppose A, B are closed subsets of $X = s_1 \times s_2$ such that $A \subset s_1 \times p_2$ and $\pi_1|_B$ is a homeomorphism of B onto A . Then there is a homeomorphism h of X onto X such that $h|_B = \pi_1|_B$.

This lemma may be proved by a repeated use of Tietze's Extension Theorem and is similar to Theorem 4.1 in [1]. We omit the proof here.

PROOF OF THE THEOREM. Let g_0 be a homeomorphism of F onto s . Denote $g_0(A)$ by A_0 , $g_0(f(A))$ by B_0 . Then $f_0 = g_0 f g_0^{-1}|_{A_0}$ is a homeomorphism of A_0 onto B_0 . By [2] there is a homeomorphism g_1 of s onto s such that $g_1(A_0 \cup B_0) \subset \bar{s}_0$ where $\bar{s}_0 = \dots \times p_{-2} \times p_{-1} \times s_0 \times p_1 \times p_2 \times \dots$ and $g_1(A_0 \cup B_0)$ has property Z in \bar{s}_0 . Denote $g_1(A_0)$ by A_1 and $g_1(B_0)$ by B_1 . Then $f_1 = g_1 f_0 g_1^{-1}|_{A_1}$ is a homeomorphism of A_1 onto B_1 . A_1 and B_1 are both closed and have property Z in \bar{s}_0 . Identifying \bar{s}_0 with s_0 , by [2] again there is an homeomorphism G of s_0 onto itself such that $G|_{A_1} = f_1$. Let K be the subset of s defined by

$$K = \{ \bar{x} = (\dots, G^{-2}(x), G^{-1}(x), x, G(x), G^2(x), \dots) \mid x \in s_0 \text{ and } G^i(x) \in s_i \}.$$

Clearly K is closed and the projection $\bar{x} \rightarrow x$ is 1-1. By the lemma there is a homeomorphism g_2 of s onto itself such that $g_2(x) = \bar{x}$ for all $x \in \bar{s}_0$. Let $\mu = g_2 g_1 g_0$. Then $\mu f \mu^{-1}|_{\mu(A)}$ is a homeomorphism of $\mu(A)$ onto $\mu(B)$. A simple verification that $\mu f \mu^{-1}|_{\mu(A)} = \Phi|_{\mu(A)}$ completes the proof of the theorem.

THEOREM 1. Let X be a separable complete metric space and let f be a homeomorphism of X onto itself. Then there is a homeomorphism μ of X into Q such that

$$\mu f \mu^{-1}|_{\mu(X)} = \Phi|_{\mu(X)}.$$

COROLLARY. Let X, f be as above, then there is a compactification μ of X into a compact subset $cl(\mu(X))$ of Q such that $\mu f \mu^{-1}|_{\mu(X)}$ extends to a homeomorphism of $cl(\mu(X))$ onto itself.

PROOF OF THEOREM 1. By the classical Banach-Mazur-Kuratowski Theorem, X can be regarded as a closed subset of the space C of continuous real valued functions on the closed unit intervals. It is well known that C can be imbedded as a closed subset of s_0 . Hence by considering $s_0 \times s_0$, we may assume that X is imbedded as a closed subset of s_0 and has property Z in s_0 . The rest of the proof is the same as the theorem.

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