RINGS HAVING SOLVABLE ADJOINT GROUPS

P. B. BHATTACHARYA AND S. K. JAIN

ABSTRACT. Let $^\circ R$ denote the group of quasi-regular elements of a ring R with respect to circle operation. The following results have been proved: (1) If R is a perfect ring and $^\circ R$ is finitely generated solvable group then R is finite and hence $^\circ R = P_1 \circ P_2 \circ \cdots \circ P_m$ where P_i are pairwise commuting p-groups. (2) Let R be a locally matrix ring or a prime ring with nonzero socle. Then $^\circ R$ is solvable iff R is either a field or a 2×2 matrix ring over a field having at most 3 elements.

For a ring R let $J(R)$ denote the Jacobson radical, $^\circ R$ the group of quasi-regular elements with respect to circle operation and $U(R)$ the group of units if R has identity. We know that if R has identity, then $^\circ R$ is isomorphic to $U(R)$. $^\circ R$ is called the adjoint group of R. The object of this paper is to study certain classes of rings R for which $^\circ R$ is nilpotent, supersolvable or solvable.

1.1. Let M be a unital free R-module over a ring R in which 2 is invertible and let $U(S)$ be supersolvable where $S = \mathrm{Hom}_R(M, M)$. Then $S = R$.

If $S \neq R$, then S contains a copy T of a 2×2 matrix ring over R. In this case we choose $a, b, c \in U(T)$ such that

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}, \quad c = \begin{pmatrix} -1 & \frac{1}{2} \\ -2 & 1 \end{pmatrix}.$$

Direct computation yields $a^{-1}b^{-1}ab = a$, $b^{-1}c^{-1}bc = b$. These relations first imply that a, b belong to the derived group of $U(T)$ and further the relation $a^{-1}b^{-1}ab = a$ implies that the derived group cannot be nilpotent. Hence $U(T)$ cannot be supersolvable in contradiction to the hypothesis that $U(S)$ is supersolvable. This proves 1.1.

The following example shows that if $2 = 0$ in a ring R then 1.1 may not be true: The group of units of a 2×2 matrix ring over a field of 2 elements is supersolvable.

1.2. Let M be a unital free module over a ring R of characteristic 2 and let the group $U(S)$ be nilpotent where $S = \mathrm{Hom}_R(M, M)$. Then $S = R$.

Received by the editors February 6, 1969 and, in revised form, November 13, 1969.

AMS 1970 *Subject Classifications*. Primary 16A26.

Key Words and Phrases. Perfect rings, locally matrix rings, prime rings with nonzero socles, solvable adjoint groups.

We argue as in 1.1. Here we choose

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Again $a^{-1}b^{-1}ab = a$ and this shows that the subgroup consisting of all the 2×2 matrices over R cannot be nilpotent, in contradiction to the hypothesis that the group $U(S)$ is nilpotent. Hence $S = R$.

1.3. (Hua). If D is a division ring and $°D$ is solvable then D is a field.

1.4. The group of units of the ring of all linear transformations of a vector space V over a division ring D is nilpotent iff $\dim_D V = 1$ and D is a field.

Since nilpotent groups are solvable the result follows from 1.2 and 1.3.

1.5. (Scott). Let F be a field, $G = \mathrm{SL}(2, F)$ and $\#(F) > 3$. Then $G = G^1$.

1.6. (Dickson). The group of quasi-regular elements in the 3×3 matrix ring over a field having at most 3 elements is not solvable.

1.7. The group of units of the ring of all linear transformations of a vector space V over a division ring D is solvable iff either D is any field and $\dim_D V = 1$ or D is a field having at most 3 elements and $\dim_D V = 2$.

This is a consequence of 1.3, 1.5 and 1.6.

1.8. If R is any ring such that $R | J(R)$ is artinian and $°R$ is solvable then $R | J(R)$ is a finite direct sum of rings R_i where R_i is a field or a 2×2 matrix ring over a field having at most 3 elements.

Since homomorphic image of a solvable group is solvable, the result follows from 1.7.

1.9. (Bass). R is a ring with dcc for principal right ideals iff $R | J(R)$ is artinian and $J(R)$ is T-nilpotent.

Such rings have been called perfect rings by Bass.

1.10. If R is a perfect ring and $°R$ is a finitely generated solvable group then R is finite, and hence $°R = P_1 \circ \cdots \circ P_m$ where P_i are pairwise commuting p-groups.

Since the fields whose multiplicative groups are finitely generated are Galois fields, finiteness of $R | J(R)$ follows at once from 1.8 and 1.9. This implies $°J(R)$ is also finitely generated. Further $J(R)$ is locally nilpotent since it is T-nilpotent. Hence $J(R)$ is nilpotent which implies $°J(R)$ is nilpotent and therefore by Watters $J(R)^+$ is finitely generated. Since R is perfect, we get $J(R)$ is finite. Hence R is finite. The last assertion is a consequence of Hall's well-known theorem for finite solvable groups.

2. We now proceed to characterise the class of prime rings, with nonzero socles, for which $^\circ R$ is solvable.

We denote by C the class of all rings which are either fields or 2×2 matrix rings over fields having at most 3 elements. In what follows it is assumed that $^\circ R$ is solvable.

2.1. A locally matrix ring R over a division ring is in C.

We recall that R is a locally matrix ring over a division ring Δ if each finite subset is in a subring of R isomorphic to an $n\times n$ matrix ring over Δ, for some natural number n. If R is not a 2×2 matrix ring over a field having at most 3 elements or it is not a field having less than 82 elements then R must contain at least 82 distinct elements a_i. These a_i can be imbedded in a subring S of R where S is an $n\times n$ matrix ring over Δ. Since $^\circ S$ is a subgroup of $^\circ R$, the subgroup $^\circ S$ is also solvable. By 1.7 we get that $S\cong\Delta$ is a commutative field. Since S contains arbitrary elements of R, we get R is commutative and hence R is a field proving that R is in C.

Since by Litoff theorem simple ring with minimal one-sided ideals is a locally matrix ring over a division ring 2.1 gives

COROLLARY. *Simple rings with minimal one-sided ideals are in the class C.*

2.2. If R is a prime ring with nonzero socle then R is in C. We know that R is then a primitive ring with nonzero socle.

The socle is a simple ring with minimal one-sided ideals. Thus by 2.1 socle (as a ring) has identity. But then the identity of the socle (which is also an ideal in R) is easily shown to be the identity for the whole ring. Hence $R=$socle, proving that R is in the class C.

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OHIO UNIVERSITY, ATHENS, OHIO 45810