GROUPS WITH AN IRREDUCIBLE CHARACTER OF LARGE DEGREE ARE SOLVABLE

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Abstract. The degree of an irreducible complex character afforded by a finite group is bounded above by the index of an abelian normal subgroup and by the square root of the index of the center. Whenever a finite group affords an irreducible character whose degree achieves these two upper bounds the group must be solvable.

Let $G$ be a finite group with an irreducible (complex) character $\xi$. If $Z$ is the center of $G$ it is easy to prove that $\xi(1)^2 \leq [G:Z]$ and if $A$ is an abelian normal subgroup of $G$ it is easy to show that $\xi(1) \leq [G:A]$ (see pp. 364–365 of [1]). Say the group $G$ admits an irreducible character of large degree if $\xi(1)^2 = [G:A]^2 = [G:Z]$, that is whenever the two bounds noted above are achieved simultaneously. Such groups arise in the theory of projective representations and the Galois theory in general rings [2]. The purpose of this note is to give proof of the result stated in the title, thus verifying a special case of a conjecture in [2]. Throughout all unexplained terminology is as in [1].

Theorem. Let $G$ be a group with center $Z$ and abelian normal subgroup $A$ so that there is an irreducible complex character $\xi$ on $G$ with $\xi(1)^2 = [G:A]^2 = [G:Z]$. Then $G$ is solvable.

Proof. A theorem of P. Hall (Theorem 4.5, p. 233 of [3]) asserts that a group is solvable if and only if every $p$-sylow subgroup has a complement. This theorem will be applied to $G/A$ to give the result.

Since the degree of any irreducible character is bounded by the index of an abelian subgroup, $A$ is a maximal abelian normal subgroup of $G$, so $Z \subseteq A$. If $\pi$, $\Pi$, and $P$ are sylow $p$-subgroups of $Z$, $A$, and $G$ respectively then $\pi \subseteq \Pi \subseteq P$. Moreover $\pi$ is contained in the center of $P$, $\Pi$ is an abelian normal subgroup of $P$, and $\Pi$ is a normal subgroup of $G$.

Arguing as in [2] we show $\xi|_P = m\lambda$ where $\lambda$ is irreducible on $P$ and $\lambda(1) = [P:\Pi]$. By Schur's lemma we have $\xi|_Z = \xi(1)\phi$ for some linear character $\phi$ of $Z$. Then $(\xi, \phi^\phi) = (\xi|_Z, \phi) = \xi(1)$ so by counting degrees $\xi(1)^2 = [G:A]$. Let $R$ be the subgroup of $G$ generated by $Z$ and $P$, and let $\lambda$ be an irreducible character of $R$ contained in $\phi^R$. By Schur's
lemma $\lambda |_P$ remains irreducible because the elements of $Z$ are represented by scalars. Since $\lambda$ is contained in $\phi^R$, $\lambda^G = m\tilde{\xi}$ for some integer $m$. By counting degrees

$$m = [G:R]\lambda(1)/\xi(1).$$

Since $\lambda$ is irreducible on $P$, $\lambda(1) = p^a$ for some $a$; $[G:R]$ is prime to $p$ since $R$ contains $P$; the $p$-part of $\xi(1)^2$ is $[P:P\cap Z]$ and $P\cap Z = \pi$. Thus we have shown $\lambda(1)^2 = [P:\pi] = (P:\Pi)^2$ and $\xi |_P = m\lambda$. Now by Cliffords theory (p. 343 of [1])

$$\lambda |_\Pi = \alpha_1 + \alpha_2 + \cdots + \alpha^n$$

where the $\alpha^i$ are conjugate linear characters on $\Pi$ (conjugate by elements in $P$). Let $\alpha = \alpha_i$ and $g \in G$, it follows that $\alpha^{(o)}$ is contained in $\lambda^{(o)} |_\Pi$ which in turn is contained in $\xi^{(o)} |_\Pi = \xi |_\Pi = m\lambda |_\Pi$. Thus $\alpha^{(o)} = \alpha_i$ for some $i$, and every $G$-conjugate of $\alpha$ is a $P$-conjugate of $\alpha$. To determine $n$ observe that the $p$-part of $\xi(1)$ is $[P:\Pi] = \lambda(1) = n$ which is the $p$-part of $[G:A]$. Let $H^* = \{g \in G | \alpha^{(o)} = \alpha\}$, then $H^*$ is the inertia group of $\alpha$ in $G$ and $[G:H^*]$ is the number of conjugates of $\alpha$ (p. 346 of [1]). Thus $[G:H^*]$ is the $p$-part of $[G:A]$ so $H^*/A$ is a $p$-complement of $P/A$ in $G/A$ which proves the theorem.

To show that the hypothesis on the center $Z$ of $G$ is necessary for the theorem let $H$ be any group and let $J_p(H)$ be the group algebra of $H$ over the field with $p$-elements ($p$ any prime). Let $A = J_p(H)$ viewed as an additive group and let $H$ act as a group of automorphisms of $A$ by

$$h(ax) = ahx \quad \text{(regular representation)} \ x, h \in H, a \in J_p.$$ 

Let $G$ be the semidirect product of $A$ by $H$ with respect to this representation of $H$ as automorphisms of $A$. Then $A$ is an abelian normal subgroup of $G$. Let $\theta$ be the linear character on $A$ defined by $\theta(ax) = \xi^a \delta_{x,1}$ where 1 is the identity in $H$, $\xi$ is a primitive $p$th root of 1, and $a$ is the least positive integer representing the corresponding class in $J_p$. It is easy to see that $\theta$ has $[H:1]$ distinct conjugates under the action of $G$ so $\theta^G$ is irreducible and $\theta(1) = [G:A]$. Observe that $H = G/A$ is arbitrary in this construction.

It is also easy to observe that if $G$ is a group satisfying all the hypotheses of the theorem and if we let $H = G/A$ then the natural semidirect product of $A$ by $H$ also satisfies the hypotheses of the theorem.

There is a central extension of $A_4 \times C_3$ ($A_4$ the alternating group of order 12, $C_3$ the cyclic group of order 3) which is a group with an irreducible character of large degree, this group is not nilpotent.
Using [2] (Theorem 1) the relationship between the groups we are studying, projective representations, and the Schur multiplier can be pointed out. If \( \alpha \) is in the Schur multiplier of the group \( G \) and \( K \) is the complex field then \((KG)_\alpha\) denotes the corresponding projective group algebra.

**Corollary.** There is a group \( H \) and an element \( \alpha \) in the Schur multiplier of \( H \) so that \((KH)_\alpha\) has center \( K \) and an abelian normal subgroup \( A \) of \( H \) with \([H:A]^2 = [H:1]\) if and only if there is a central extension \( G \) of \( H \) satisfying the hypothesis of the theorem.

**Proof.** The corollary is immediate on combining the theorem and Theorem 1 of [2].

**Bibliography**


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