LOCALLY COMPACT SPACES OF MEASURES

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ABSTRACT. Under varying conditions on the topological space X, the spaces of σ-smooth, τ-smooth, and tight measures on X, respectively, are each shown to be locally compact in the weak topology if, and only if, X is compact.

1. **Introduction.** Using the methods and results of Varadarajan [5], we give necessary and sufficient conditions on the topological space X for the local compactness of the spaces of σ-smooth, τ-smooth, and tight measures [resp., signed measures] on X, respectively, when endowed with the weak topology. For such spaces of signed measures, the finiteness of X is easily seen to be necessary and sufficient (Remark 4). For such spaces of (nonnegative) measures, we find that, under various conditions on X, the compactness of X is necessary and sufficient (Corollaries 3 and 4); in fact, it is the dividing line between when the space is locally compact and when it has no compact neighborhoods whatsoever (Theorems 1, 2, and 5).

2. **Preliminaries and main results.** Our definitions, notation, and terminology follow quite closely those in the paper of Varadarajan [5]. In particular, our topological definitions and terminology coincide with those of Kelley [3]. X will denote a topological space and $C(X)$ the space of all real-valued, bounded, continuous functions on X. We use $\mathfrak{Z}$ to denote the class of all *zero-sets* on X; i.e., $\mathfrak{Z} = \{f^{-1}(0); f \in C(X)\}$. The class of all *co-zero* sets (i.e., complements of zero-sets) in X shall be denoted by $\mathfrak{U}$. The smallest algebra which contains $\mathfrak{Z}$ shall be denoted by $\mathfrak{F}$.

A *measure* will mean a nonnegative, finite, finitely additive set function m on $\mathfrak{F}$ such that for every $A \in \mathfrak{F}$,

$$m(A) = \sup\{m(Z); A \supset Z \in \mathfrak{Z}\}.$$

The difference between any two measures is called a *signed measure*.

We shall let $\mathfrak{M}(X) = \{m; m \text{ is a signed measure}\}$ and $\mathfrak{M}^+(X) = \{m \in \mathfrak{M}(X); m \geq 0\}$. We define, for each $m \in \mathfrak{M}(X)$, the norm

$\|m\| = |m|(X) = m^+(X) + m^-(X)$, where m^+, m^-, and $|m|$ denote the positive, negative, and total variations of m, respectively. $\mathfrak{M}(X)$ with this norm is a Banach space. However, we shall use this norm only for descriptive purposes since we are interested in the weak topology (called the weak*-topology by some authors) on $\mathfrak{M}(X)$; this is the topology with basic neighborhoods of the form

$$N(m_0; f_1, \cdots, f_n, \epsilon)$$

$$= \left\{ m \in \mathfrak{M}(X); \left| \int f_i dm - \int f_i dm_0 \right| < \epsilon, \ i = 1, \cdots, n \right\}$$

where $m_0 \in \mathfrak{M}(X)$, $\{f_1, \cdots, f_n\}$ is any finite subset of $C(X)$, and $\epsilon > 0$. A net $\{m_\alpha\}$ in $\mathfrak{M}(X)$ converges to $m \in \mathfrak{M}(X)$ in the weak topology if, and only if, $\int f dm_\alpha \to \int f dm$ for every $f \in C(X)$. $\mathfrak{M}(X)$ with the weak topology is a completely regular, Hausdorff, locally convex linear topological space and $\mathfrak{M}^+(X)$ is a closed subset [5, Theorem 1, p. 181].

We shall be especially interested in certain subspaces of $\mathfrak{M}(X)$ endowed with the relativized weak topology. We describe them now. Let $m \in \mathfrak{M}(X)$. m is σ-*smooth* if, and only if, $m(A_n) \to 0$ whenever $\{A_n\}$ is a sequence of sets in $\mathfrak{F}$ which decreases to $\varnothing$. m is τ-*smooth* if, and only if, $|m|(Z_\alpha) \to 0$ for any net $\{Z_\alpha\}$ of zero-sets for which (i) $Z_\beta \subset Z_\alpha$ if $\beta > \alpha$ and (ii) $\cap Z_\alpha = \varnothing$. Finally, m is *tight* if, and only if, for any $\delta > 0$ there exists a compact set $K \subset X$ such that $|m|_*(X - K) < \delta$ where $|m|_*$ is defined for each $E \subset X$ by

$$|m|_*(E) = \sup\{ |m|(Z); E \supset Z \in \mathfrak{F} \}.$$

(The definitions of τ-smooth and tight here are not the same as in Varadarajan [5], but are equivalent [5, Theorem 24, p. 174, Theorem 29, p. 179].) We shall use $\mathfrak{M}_\sigma(X)$, $\mathfrak{M}_\tau(X)$, and $\mathfrak{M}_t(X)$ to denote the sets of σ-smooth, τ-smooth, and tight signed measures, respectively; $\mathfrak{M}_\sigma^+(X)$, $\mathfrak{M}_\tau^+(X)$, and $\mathfrak{M}_t^+(X)$ will denote the spaces of σ-smooth, τ-smooth, and tight measures, respectively. It is not hard to see that $\mathfrak{M}_t(X) \subset \mathfrak{M}_\tau(X) \subset \mathfrak{M}_\sigma(X) \subset \mathfrak{M}(X)$; hence also $\mathfrak{M}_t^+(X) \subset \mathfrak{M}_\tau^+(X) \subset \mathfrak{M}_\sigma^+(X) \subset \mathfrak{M}^+(X)$. We note that $\mathfrak{M}_\sigma^+(X)$ is a closed subset of $\mathfrak{M}_\sigma(X)$ in the weak topology and that the same is true if σ is replaced by τ or t.

A measure $m \in \mathfrak{M}^+(X)$ is *two-valued* if, and only if, (i) $m \neq 0$ and (ii) $m(A)$ is 0 or 1 for every $A \in \mathfrak{F}$. For each $x \in X$, we let p_x denote the two-valued measure degenerate at the point x; i.e.,

$$p_x(A) = 1, \quad x \in A \atop = 0, \quad x \notin A \qquad \text{for every } A \in \mathfrak{F}.$$

We let $\mathfrak{T}^+(X) = \{m \in \mathfrak{M}^+(X); \; m$ is two-valued$\}$ and $\mathfrak{D}(X) = \{p_x; \; x \in X\}$. Clearly $\mathfrak{D}(X) \subset \mathfrak{M}_t^+(X)$ and $\mathfrak{D}(X) \subset \mathfrak{T}^+(X)$.

Each of the following remarks is either well known or easily verified.

REMARKS. 1. The norm-bounded subsets of $\mathfrak{M}(X)$ are relatively compact (i.e., their closures are compact) in the weak topology.

2. $\mathfrak{M}(X)$ is σ-compact and $\mathfrak{M}^+(X)$ is locally compact and σ-compact. This follows easily from Remark 1 and the fact that $\mathfrak{M}^+(X)$ is closed in $\mathfrak{M}(X)$.

3. If X is compact, then $\mathfrak{M}(X) = \mathfrak{M}_t(X)$ ($= \mathfrak{M}_\sigma(X) = \mathfrak{M}_r(X)$). The converse is true if X is completely regular and Hausdorff.

4. If X is a completely regular, Hausdorff space, then the finiteness of X is necessary and sufficient for the local compactness of any one of $\mathfrak{M}(X)$, $\mathfrak{M}_\sigma(X)$, $\mathfrak{M}_r(X)$, or $\mathfrak{M}_t(X)$. (For necessity, see [4, Theorem 7.8]; sufficiency follows from Remarks 3 and 1.)

5. If X is a completely regular, Hausdorff space, then

(i) $\mathfrak{T}^+(X)$ is closed in $\mathfrak{M}(X)$,

(ii) $\mathfrak{T}^+(X) \cap \mathfrak{M}_r^+(X) = \mathfrak{T}^+(X) \cap \mathfrak{M}_t^+(X) = \mathfrak{D}(X)$, and

(iii) $T(x) = p_x$ defines a homeomorphism of X onto $\mathfrak{D}(X)$ [5, Theorem 11, p. 187, Corollary 3, p. 175, Theorem 9, p. 187].

6. A set $F \subset \mathfrak{M}_\sigma^+(X)$ is relatively compact in $\mathfrak{M}_\sigma^+(X)$ if, and only if, for any sequence $\{g_j\}$ in $C(X)$ which decreases to zero, $\int g_j dm \to 0$ uniformly in $m \in F$ [5, Theorem 25(2), p. 200].

We shall also use the fact [5, Remark 3, p. 183] that the family of all sets of the form

$$N^*(m_0; U_1, \cdots, U_k, \epsilon)$$
$$= \{m \in \mathfrak{M}^+(X); m(U_i) - m_0(U_i) > -\epsilon$$
$$\text{for all } i = 1, \cdots, k, \quad \text{and} \quad |m(X) - m_0(X)| < \epsilon\}$$

where $m_0 \in \mathfrak{M}^+(X)$, $U_1, \cdots, U_k \in \mathfrak{U}$, and $\epsilon > 0$ form a basis for the neighborhoods of the weak topology on $\mathfrak{M}^+(X)$.

We shall now state and prove our main results. We recall that any metric space is paracompact.

THEOREM 1. *Let X be a paracompact, Hausdorff topological space.*

(I) *If X is compact, then $\mathfrak{M}_\sigma^+(X)$ is locally compact and σ-compact.*

(II) *If X is not compact, then $\mathfrak{M}_\sigma^+(X)$ has no compact neighborhoods whatsoever.*

REMARK. We can weaken our hypothesis on X in Theorem 1 (and in Corollary 4 below) to metacompact, normal, and Hausdorff (see [3, 5V(c)]) or to realcompact, normal, and Hausdorff.

PROOF. (I) follows from Remarks 2 and 3. For the proof of (II), we let $m_0 \in \mathfrak{M}_\sigma^+(X)$ and consider a basic neighborhood

$$V = N^*(m_0; U_1, \cdots, U_k, \epsilon) \cap \mathfrak{M}_\sigma^+(X).$$

There is a sequence $\{x_n\} \subset X$ which has no cluster point since X is paracompact, Hausdorff and is not compact. Let $m_n = m_0 + (\epsilon/2)p_{x_n}$, $n = 1, 2, \cdots$. Then clearly $\{m_n\} \subset V$. To show V is not relatively compact in $\mathfrak{M}_\sigma^+(X)$, it suffices to show that $\{m_n\}$ is not relatively compact in $\mathfrak{M}_\sigma^+(X)$.

We define $F_n = \{x_m; m \neq n\}$, $n = 1, 2, \cdots$. Since $\{x_n\}$ has no cluster point, F_n is closed for every n. Consequently, by the normality of X there are open sets U_n and closed sets Y_n such that $x_n \in U_n \subset Y_n \subset X - F_n$ for every n. Define $V_n = U_n - \bigcup_{i=1}^{n-1} Y_i$, $n = 1, 2, \cdots$. The open sets V_n are disjoint and $x_n \in V_n$ for every n. We define $H_k = X - \bigcup_{n \geq k} V_n$, $k = 1, 2, \cdots$. Then $\{H_k\}$ is an increasing sequence of closed sets whose union is X. Moreover, H_k and $\{x_k, x_{k+1}, \cdots\}$ are disjoint closed sets so that, by the normality of X, there are continuous functions $h_k : X \to [0, 1]$ such that h_k is 0 on H_k and is 1 on $\{x_k, x_{k+1}, \cdots\}$. If we define $g_j = \min\{h_1, \cdots, h_j\}$, $j = 1, 2, \cdots$, one can readily see that $g_j \in C(X)$, g_j decreases to the zero function, and $g_j = 1$ on $\{x_j, x_{j+1}, \cdots\}$. But, consequently, if $n \geq j$, then $\int g_j dm_n \geq \epsilon/2$. Hence by Remark 6 $\{m_n\}$ is not relatively compact.

EXAMPLE. If X is pseudo-compact (i.e., every continuous function on X is bounded), then $\mathfrak{M}_\sigma(X) = \mathfrak{M}(X)$ ([1], [5, Theorem 21, p. 172]) and, consequently, $\mathfrak{M}_\sigma^+(X)$ is locally compact (Remark 2). Since there is a countably compact (hence pseudo-compact), locally compact, first countable, normal, Hausdorff space which is not compact [3, 5E(e)], we cannot hope, in either Theorem 1 or Corollary 4 (below), to significantly weaken the hypotheses on X beyond that remarked above.

THEOREM 2. *Let X be a completely regular, Hausdorff space.*

(I) *If X is compact, then $\mathfrak{M}_t^+(X)$ ($= \mathfrak{M}_\tau^+(X)$) is locally compact and σ-compact.*

(II) *If X is not compact, then neither $\mathfrak{M}_t^+(X)$ nor $\mathfrak{M}_\tau^+(X)$ has any compact neighborhoods whatsoever.*

PROOF. The proof of (II) is the same for $\mathfrak{M}_t^+(X)$ and $\mathfrak{M}_\tau^+(X)$ so we restrict ourselves to the latter case. Let $m_0 \in \mathfrak{M}_t^+(X)$ and consider $V = N^*(m_0; U_1, \cdots, U_k, \epsilon) \cap \mathfrak{M}_t^+(X)$. Since X is not compact there is a net $\{x_\alpha\}_{\alpha \in A}$ in X which has no cluster point in X. Define $m_\alpha = m_0 + (\epsilon/2)p_{x_\alpha}$ for every $\alpha \in A$. Clearly $\{m_\alpha\}_{\alpha \in A} \subset V$. Thus it suffices to show that the closure of $\{m_\alpha\}_{\alpha \in A}$ in $\mathfrak{M}_t^+(X)$ is not compact. This set *is* compact if, and only if, the closure of $\{p_{x_\alpha}\}_{\alpha \in A}$ in $\mathfrak{M}_t^+(X)$ is compact. But this, in turn, is equivalent to the compactness of the closure of $\{p_{x_\alpha}\}_{\alpha \in A}$ in $\mathfrak{D}(X)$ because $\mathfrak{D}(X)$ is closed in $\mathfrak{M}_t^+(X)$ (Remark 5

(i), (ii)). Equivalently, by Remark 5 (iii), the closure of $\{x_\alpha\}_{\alpha\in A}$ in X is compact. But this leads to a contradiction since $\{x_\alpha\}_{\alpha\in A}$ has no cluster point in X.

COROLLARY 3. *Let X be completely regular and Hausdorff. The following are equivalent*:
(a) *X is compact.*
(b) *$\mathfrak{M}_\tau^+(X)$ is locally compact.*
(c) *$\mathfrak{M}_t^+(X)$ is locally compact.*

The preceding corollary and Theorem 1 yield:

COROLLARY 4. *Let X be a paracompact, Hausdorff space. The following are equivalent*:
(a) *X is compact.*
(b) *$\mathfrak{M}_\sigma^+(X)$ is locally compact.*
(c) *$\mathfrak{M}_\tau^+(X)$ is locally compact.*
(d) *$\mathfrak{M}_t^+(X)$ is locally compact.*

The following result is contained in Theorem 2 if X is Hausdorff. We omit the proof which begins in the same manner as that of Theorem 2 and ends with the use of [5, Theorem 29, p. 205].

THEOREM 5. *Let X be a locally compact, regular space such that $\mathfrak{M}_\sigma^+(X) = \mathfrak{M}_t^+(X)$.*
(I) *If X is compact, then $\mathfrak{M}_\sigma^+(X)$ $(=\mathfrak{M}_\tau^+(X)=\mathfrak{M}_t^+(X))$ is locally compact and σ-compact.*
(II) *If X is not compact, then $\mathfrak{M}_\sigma^+(X)$ $(=\mathfrak{M}_\tau^+(X)=\mathfrak{M}_t^+(X))$ has no compact neighborhoods whatsoever.*

REMARK. Varadarajan [5, Remark, p. 205] gives conditions under which the hypotheses of Theorem 5 are fulfilled. One instance is when X is locally compact, regular, and σ-compact.

EXAMPLE. Let X be a regular, Hausdorff, noncompact space such that every real-valued continuous function on X is constant [2]. Clearly $\mathfrak{M}(X)=\mathfrak{M}_\sigma(X)=\mathfrak{M}_\tau(X)=\mathfrak{M}_t(X)$ is locally compact, thus so is $\mathfrak{M}^+(X)=\mathfrak{M}_\sigma^+(X)=\mathfrak{M}_\tau^+(X)=\mathfrak{M}_t^+(X)$. Consequently, we cannot hope to significantly weaken the hypotheses on X in Remark 4, in Theorems 2 or 5, or in Corollary 3.

3. **Complements; probability measures.** Let $P_\sigma(X) = \{m\in\mathfrak{M}_\sigma^+(X);\ m(X)=1\}$, $P_\tau(X) = \{m\in\mathfrak{M}_\tau^+(X);\ m(X)=1\}$, and $P_t(X) = \{m\in\mathfrak{M}_t^+(X);\ m(X)=1\}$, each endowed with the relativized weak topology. Theorems 1, 2, and 5 and Corollaries 3 and 4 all remain valid if $\mathfrak{M}_\sigma^+(X)$, $\mathfrak{M}_\tau^+(X)$, and $\mathfrak{M}_t^+(X)$ are replaced by $P_\sigma(X)$, $P_\tau(X)$, and $P_t(X)$, respectively. For in the proof of Theorem 1, along with

replacing $\mathfrak{M}_\sigma^+(X)$ by $P_\sigma(X)$, one must simply change the definition of m_n to $m_n = (1-\epsilon/2)m_0 + (\epsilon/2)p_{x_n}$ $(0 < \epsilon \leq 2)$; and similarly in the proofs of Theorems 2 and 5. Then these proofs will apply since $P_\sigma(X)$, $P_\tau(X)$, and $P_t(X)$ are closed in $\mathfrak{M}_\sigma^+(X)$, $\mathfrak{M}_\tau^+(X)$, and $\mathfrak{M}_t^+(X)$, respectively.

Now if X is a completely regular, Hausdorff space, then $\mathfrak{D}(X)$ is closed in each of $\mathfrak{M}_\tau^+(X)$, $\mathfrak{M}_t^+(X)$, their respective closed unit balls, $P_\tau(X)$, and $P_t(X)$. Hence X will share any topological properties of these spaces which are inherited by closed subspaces (Remark 5 (iii)). Normality, paracompactness, compactness, and σ-compactness are examples. This observation verifies the implications (b)$\Rightarrow$(a), (c)$\Rightarrow$(a), (d)$\Rightarrow$(a), and (e)$\Rightarrow$(a) of our final result, which must be well known. The reverse implications follow from Remarks 3 and 1.

THEOREM 6. *Let X be a completely regular, Hausdorff space. The following are equivalent*:

(a) *X is compact.*

(b) *The closed unit ball $\{m \in \mathfrak{M}_\tau^+(X); \|m\| \leq 1\}$ in $\mathfrak{M}_\tau^+(X)$ is compact.*

(c) *The closed unit ball $\{m \in \mathfrak{M}_t^+(X); \|m\| \leq 1\}$ in $\mathfrak{M}_t^+(X)$ is compact.*

(d) *$P_\tau(X)$ is compact.*

(e) *$P_t(X)$ is compact.*

It follows that if X is a completely regular, Hausdorff space, then $P_\tau(X)$ [resp., $P_t(X)$] is locally compact if, and only if, it is compact. Clearly the same result holds for $P_\sigma(X)$ if X is a normal, Hausdorff space.

4. Added in proof. The following result generalizes Theorem 1.

THEOREM 1′. *Let X be a topological space.*

(I) *If X is pseudo-compact, then $\mathfrak{M}_\sigma^+(X)$ is locally compact and σ-compact.*

(II) *If X is not pseudo-compact, then $\mathfrak{M}_\sigma^+(X)$ has no compact neighborhoods whatsoever.*

PROOF. (I) follows from Remark 2. For (II) we let $m_0 \in \mathfrak{M}_\sigma^+(X)$ and consider $V = N^*(m_0, U_1, \cdots, U_k, \epsilon) \cap \mathfrak{M}_\sigma^+(X)$. Since X is not pseudo-compact, there is a regular sequence $\{Z_n\}$ of zero-sets in X such that $X - Z_n \neq \varnothing$ for every n [5, Theorem 16, p. 170]. Let $x_n \in X - Z_n$ and define $m_n = m_0 + (\epsilon/2)p_{x_n}$, $n = 1, 2, \cdots$. Then clearly $\{m_n\} \subset V$. But $m_n(X - Z_n) \geq \epsilon/2$ for every n so that V is not relatively compact [5, Theorem 25 (1), p. 200].

As before, Theorem 1′ remains valid if $\mathfrak{M}_\sigma^+(X)$ is replaced by $P_\sigma(X)$. Consequently, if X is *any* topological space, then $P_\sigma(X)$ is locally compact if, and only if, it is compact.

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