INJECTIVE ENDOMORPHISMS OF FINITELY GENERATED MODULES

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ABSTRACT. Let R be a commutative ring. Then any injective endomorphism of a finitely generated R-module is always an isomorphism if and only if R is 0-dimensional, that is, if every prime ideal is maximal.

This note aims at considering cases where an injective endomorphism of a finitely generated module is, actually, an isomorphism. It is a simple exercise that artinian modules are endowed with this property [**1**, p. 23] and here we will show that the commutative rings for which the fact above is always true resemble artinian rings. A similar question on when surjective endomorphisms of finitely generated modules are isomorphisms was proved independently by Strooker [**3**] and the author [**4**] or [**5**] for any commutative ring, regardless of finite presentation [**2**, p. 35] or chain conditions [**1**, p. 23].

For a commutative ring R, the result of this note says

THEOREM. *Any injective endomorphism of a finitely generated R-module is an isomorphism if and only if every prime ideal of R is maximal.*

That the above condition is necessary, it is easy to see: If $P \subset Q$ are two distinct primes in R, then any element $x \in Q-P$ induces, via multiplication, an injection of R/P which is not surjective. The converse takes longer to prove but it is just as easy.

Consider thus a ring R with the aforementioned property, that is, of having Krull dimension 0, and let f be an injection of the finitely generated R-module M. This module can be made into an $R[x]$-module by defining $x \cdot m = f(m)$ for $m \in M$. We claim that as an $R[x]$-module M has an annihilator I, big enough, so that $S = R[x]/I$ is 0-dimensional. To see this, let $m_1, \cdots, m_n$ be a generating set for M as an R-module. We have $xm_i = \sum r_{ij} m_j$, with $r_{ij} \in R$, that is, a system of equations

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$$r_{1i}m_1 + \cdots + (r_{ii} - x)m_i + \cdots + r_{in}m_n = 0, \qquad i = 1, \cdots, n,$$

and thus det $\alpha \cdot M = (0)$, with α the matrix of coefficients in $R[x]$. This says that I contains the monic polynomial $p(x) = \det \alpha$, and, in particular, that $S_0 = R[x]/(p(x))$ is integral over R and so also 0-dimensional, by Cohen-Seidenberg's theorem. S being a homomorphic image of S_0, also has Krull dimension 0.

Denote by u the image of x in S. If u is a unit, we are through; if not, J = annihilator of $u \neq (0)$. [Proof: if $u \in P$ = prime ideal in S, the image of u in the localization S_P is a nilpotent element and so, easily, $J \neq (0)$.] In the last case $uJM = (0)$ implies $JM = (0)$ and hence $J = (0)$, as M is S-faithful. This ends the proof.

REMARK. This result completes the chapter of when are injective or surjective endomorphisms of general finitely generated modules over commutative rings isomorphisms. It would be of obvious interest: further study of this question for restricted classes of finitely generated modules or, more significantly, its recasting in the context of noncommutative rings. In particular, the theorem of this note should be valid for rings which are close to being artinian, as for instance, perfect rings.

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