CLOSED BAIRE SETS ARE (SOMETIMES) ZERO-SETS

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ABSTRACT. It is a theorem essentially due to Paul Halmos [H, 51.D] that each compact Baire set is a zero-set. Kenneth A. Ross and Karl Stromberg [RS] have shown (a bit more than the fact that) if X is a completely regular Hausdorff space which is locally compact and σ-compact, then each closed Baire set in X is a zero-set; the same conclusion is known to hold in case X is Lindelöf and a G_δ in βX. In the present paper we prove the following theorem, and we show how the "closed Baire set" theorems of Ross and Stromberg emerge as corollaries: If X is Baire in βX and A is a closed Baire set in X, then A is a zero-set in X. Finally, we indicate how our theorem, and hence those of Ross and Stromberg, can be derived from early and forthcoming work of Frolík.

1. **Closed Baire sets.** For each family $\mathcal{A}$ of sets, the symbol $\tau(\mathcal{A})$ denotes the smallest family $\mathcal{F}$ of sets containing $\mathcal{A}$ for which $\bigcup_{n\in N} A_n$ belongs to $\mathcal{F}$ and $\bigcap_{n\in N} A_n$ belongs to $\mathcal{F}$ whenever each of the sets A_n belongs to $\mathcal{F}$ (here and throughout, the set N is the set of positive integers). The family $\tau(\mathcal{A})$ can be constructed by hand: one has

$$\tau(\mathcal{A}) = \bigcup_{0\leq\alpha<\omega_1} \mathcal{A}_\alpha,$$

where by definition

$\mathcal{A}_0 = \mathcal{A}$;

$\mathcal{A}_\alpha = \{\bigcup_{n\in N} A_n : A_n \in \mathcal{A}_{\alpha-1}\}$ for each odd nonlimit ordinal α;

$\mathcal{A}_\alpha = \{\bigcap_{n\in N} A_n : A_n \in \mathcal{A}_{\alpha-1}\}$ for each even nonlimit ordinal α;

$\mathcal{A}_\alpha = \bigcup_{\gamma<\alpha} \mathcal{A}_\gamma$ for each limit ordinal α.

(Among the predecessors of a given infinite nonlimit ordinal α, there is a largest limit ordinal $\lambda(\alpha)$. Thus α has the form $\alpha = \lambda(\alpha) + n(\alpha)$ for some positive integer $n(\alpha)$, and α is said to be odd or even according as $n(\alpha)$ is odd or even.)

We shall consider only completely regular Hausdorff spaces. For each such space X, we denote by $\mathcal{Z}(X)$, or by $\mathcal{Z}$ when confusion is impossible, the family of zero-sets in X. The family $\mathcal{B}(X)$ of Baire sets in X is, by definition, the smallest family of subsets of X containing $\mathcal{Z}$ and closed under passage to the complement, and to the countable union, and to the countable intersection, of its members.

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In the notation of the last paragraph, we have $X\backslash Z\in\mathcal{Z}_1$ whenever $Z\in\mathcal{Z}$, whence it follows in fact that $\tau(\mathcal{Z}(X))$ is closed under complementation. Thus $\mathcal{B}(X)=\tau(\mathcal{Z}(X))$.

The lemma which follows uses the technique introduced by Halmos [**H**, 51.D]; see also [$\mathbf{N}_1$, Theorem 2.1], and [$\mathbf{N}_2$, p. 607].

1.1. Lemma. *Let S and T be Baire sets in Y. Then there is a countable collection $\mathcal{W}$ of zero-sets of Y, and a continuous mapping q from Y onto some metric space M, for which*

(a) *$S\in\tau(\mathcal{W})$ and $T\in\tau(\mathcal{W})$;*

(b) *$A=q^{-1}(q(A))$ whenever $A\in\tau(\mathcal{W})$.*

Proof. Since $\tau\mathcal{A}=\bigcup\{\tau\mathcal{W}:\mathcal{W}\subset\mathcal{A}$ and $|\mathcal{W}|\leqq\aleph_0\}$ for each class $\mathcal{A}$ of sets, we can find countable subfamilies $\mathcal{W}^{(1)}$ and $\mathcal{W}^{(2)}$ of $\mathcal{Z}(Y)$ with $S\in\tau(\mathcal{W}^{(1)})$ and $T\in\tau(\mathcal{W}^{(2)})$. We write $\mathcal{W}=\mathcal{W}^{(1)}\cup\mathcal{W}^{(2)}$, and we write

$$\mathcal{W} = \{\overset{-1}{f_n}(0):n\in N\}$$

with each f_n a continuous real-valued function on Y for which $0\leqq f_n\leqq 1$.

Defining $d(x,y)=\sum_{n\in N}|f_n(x)-f_n(y)|/2^n$ for $(x,y)\in Y\times Y$, and writing $\bar{y}=\{x\in Y:d(x,y)=0\}$, the desired metric space is the set $M=\{\bar{y}:y\in Y\}$ together with the metric $\bar{d}$ defined by the rule $\bar{d}(\bar{x},\bar{y})=d(x,y)$.

That the map q defined from Y to M by the rule $q(y)=\bar{y}$ is continuous at a fixed point y_0 is nearly obvious: Given $\epsilon>0$ there is an integer m for which $\sum_{n>m}1/2^n<\epsilon/2$ and a neighborhood U of y_0 for which

$$|f_n(y)-f_n(y_0)| < \epsilon/m$$

whenever $y\in U$ and $1\leqq n\leqq m$; then

$$\begin{aligned}\bar{d}(q(y),q(y_0)) &= \sum_{1\leqq n\leqq m}|f_n(y)-f_n(y_0)|/2^n+\sum_{n>m}|f_n(y)-f_n(y_0)|/2^n\\ &\leqq m\cdot 1/2\cdot\epsilon/m+\epsilon/2=\epsilon.\end{aligned}$$

If $A\in\mathcal{W}_0=\mathcal{W}$ then, because for each pair (x,y) of points in Y, we have $q(x)=q(y)$ if and only if $f_n(x)=f_n(y)$ for each n, evidently $A=q^{-1}(q(A))$. Now suppose that $\alpha<\omega_1$ and that $B=q^{-1}(q(B))$ whenever $B\in\bigcup_{\gamma<\alpha}\mathcal{W}_\gamma$, and let $A\in\mathcal{W}_\alpha$. If α is a limit ordinal we have $A\in\mathcal{W}_\gamma$ for some $\gamma<\alpha$, so that $A=q^{-1}(q(A))$; otherwise there are elements $\{A_n\}_{n\in N}$ in $\mathcal{W}_{\alpha-1}$ for which $A=\bigcup_{n\in N}A_n$ or $A=\bigcap_{n\in N}A_n$, and in either case from the inductive hypothesis $A_n=q^{-1}(q(A))$ it follows that $A=q^{-1}(q(A))$.

We are now ready to prove the theorem cited earlier.

1.2. THEOREM. *Let X be a Baire set in βX and let A be a closed Baire set in X. Then A is a zero-set in X.*

PROOF. Because each bounded real-valued continuous function on X extends continuously to βX, there is a Baire set A' in βX for which $A = A' \cap X$; thus A itself is a Baire set in βX. From the lemma above there is a continuous map q from βX onto some metric space M with $A = q^{-1}(q(A))$ and $X = q^{-1}(q(X))$. We set $M' = q(X)$, and we denote by q' the restriction of q to X.

To see that $q'(A)$ is closed in M', let m_α be a net in $q'(A)$—say $m_\alpha = q'(x_\alpha)$ with $x_\alpha \in A$—for which $m_\alpha \to m \in M'$. Because βX is compact some subnet of the net x_α in βX converges to some point p in βX. Since q is continuous, we have $q(p) = m$, so $p \in q^{-1}(M') = q^{-1}(q(X)) = X$. Since A is closed in X we have $p \in A$, so indeed $m = q(p) \in q(A)$ and $q(A)$ is closed in M'.

Choosing any continuous real-valued function f on M' for which $q(A) = f^{-1}(0)$, we have (as in [**H**, 51.D] and elsewhere): $f \circ q'$ is continuous on X, and $A = (f \circ q')^{-1}(0)$.

1.3. COROLLARY. *Let A be a closed Baire set in X, and suppose either that*

(a) *X is locally compact and σ-compact*; or

(b) *X is locally compact and paracompact*; or

(c) *X is Lindelöf, and a G_δ in βX.*

Then A is a zero-set in X.

PROOF. (a) The locally compact, σ-compact spaces are readily checked to be precisely those spaces X which are cozero-sets in βX [**GJ**, 3.11(b)]. Each such space is a Baire set in βX, then, so 1.2 applies.

(b) follows from (a): X can be expressed in the form $X = \bigcup_{\gamma \in \Gamma} X_\gamma$, where the sets X_γ are pairwise disjoint open-and-closed σ-compact subsets of X; according to (a) there is for each γ a continuous real-valued function f_γ on X_γ for which $A \cap X_\gamma = f_\gamma^{-1}(0)$; writing $f = \bigcup_{\gamma \in \Gamma} f_\gamma$, we see that f is continuous on X and that $A = f^{-1}(0)$.

(c) Let the Lindelöf space X be expressed in the form $X = \bigcap_{n \in N} U_n$, each U_n open in βX. For each point x in X and each n there is a cozero-set $G(x, n)$ in βX with

$$x \in G(x, n) \subset U_n.$$

For n fixed there are countably many such sets whose union contains

X. Since in any space the countable union of cozero-sets is a cozero-set, we have: for each n a cozero-set G_n exists for which $X \subset G_n \subset U_n$. It follows that $X = \bigcap_{n \in N} G_n$, so that X is a Baire set in βX.

It is readily checked that a space X is locally compact if and only if X is open in βX. This fact yields the following easy result.

1.4. PROPOSITION. *Let A be a closed, Baire set in the locally compact space X. Then the following conditions are equivalent:*

(a) *A is σ-compact;*

(b) *A is a cozero-set in βA;*

(c) *A is a Baire set in βA.*

PROOF. Only the implication (c)$\Rightarrow$(a) is not obvious. If (c) holds then A, being locally compact, is an open Baire set in βA, hence is a cozero-set in βA, hence is σ-compact.

The proposition just given sheds no light on the following question, posed by Ross and Stromberg in [**RS**]: Must each closed Baire set in a locally compact, normal space be a zero-set? We do not know the answer to this question. The following result is a poor substitute.

1.5. PROPOSITION. *Let X be locally compact and normal, and let A be a closed Baire set with a locally compact, paracompact neighborhood in X. Then A is a zero-set in X.*

PROOF. This follows from 1.3(b), which coincides with Theorem 1.3 of [**RS**]. Using the normality hypothesis there are open sets U and V, with V paracompact, for which

$$A \subset U \subset \mathrm{cl}_X\, U \subset V,$$

and a continuous function f from V to $[0, 1]$ with $A = f^{-1}(0)$. Extending f continuously to a function f' from X to $[0, 1]$ and choosing a continuous function g from X to $[0, 1]$ with $g \equiv 0$ on A and $g \equiv 1$ on $X \backslash V$, we have $A = (f' + g)^{-1}(0)$, as desired.

We remark that Theorem 1.5 of [**RS**], according to which each closed, σ-compact Baire set in a normal, locally compact space is a zero-set, follows from 1.5 above; for in general, any σ-compact subset of a locally compact space admits a σ-compact (hence, paracompact) neighborhood.

2. **Contributions of Z. Frolík.** Theorem 1.2 above, and most of the results we have cited from [**RS**] (though not the [**RS**] results relating to measure and topological groups), can be deduced formally from the deep results pertaining to analytic sets codified and extended

by Frolík in [$\mathbf{F}_2$] and in a sequence of papers summarized in [$\mathbf{F}_1$]. Let us say, with Frolík, that the space X is analytic if there is an upper semicontinuous multi-valued map f from the space Σ of irrational numbers onto X which is compact (in the sense that $f(\sigma)$ is compact whenever $\sigma \in \Sigma$). Then each compact space, and each closed subspace of an analytic space, is analytic; thus each Baire set in an analytic space is analytic, hence Lindelöf. Now to prove Theorem 1.2, let A be a closed Baire set in X, where X is a Baire set in βX. Then X is analytic, and $X\backslash A$ is a Baire set in X, hence is Lindelöf; like any open Lindelöf set in any space, $X\backslash A$ is a cozero-set (this uses the fact that the countable union of cozero-sets is a cozero-set). Thus A is a zero-set in X.

Frolík has pointed out to the author, in connection with an early version of the present paper, that the proof in §1 of 1.2 carries over directly to those sets which are (in his terminology) distinguishable from their complements. A formal definition runs as follows.

2.1. Definition. The subset A of X is said to be distinguishable in X if there is a continuous function ϕ from X into some separable metric space for which

$$\phi(A) \cap \phi(X\backslash A) = \varnothing.$$

The stronger version of 1.2 is, then, as follows.

2.2. Theorem. *Let A be a closed and distinguishable subset of X, and let X be distinguishable in βX. Then A is a zero-set in X.*

For the proof, one shows as before that A is distinguishable in βX, so that there is a continuous function q from βX to some metric space M for which $q(A)\cap q(\beta X\backslash A)=\varnothing$. (The pair (q, M) is constructed as follows. There are separable metric spaces M_1 and M_2, and continuous functions ϕ and ψ from X and βX into M_1 and M_2 respectively, for which $\phi(A)\cap\phi(X\backslash A)=\psi(X)\cap\psi(\beta X\backslash X)=\varnothing$. Denoting by M_1^* a metrizable compactification of M_1 and by ϕ^* the continuous extension of ϕ mapping βX into M_1^*, we let $M=M_1^*\times M_2$ and $q(p) =(\phi^*(p), \psi(p))$ for p in βX.) As above $q(A)$ is closed in M, hence has the form $f^{-1}(0)$ for some continuous, real-valued function on M, so that $A=(f\circ q)^{-1}(0)$, as desired.

In informal conversation, Frolík has suggested the likelihood that the concept of a distinguishable set may in the long run prove at least as fruitful and productive as the familiar Baire set concept. His forthcoming paper [$\mathbf{F}_2$] may serve to substantiate this suggestion.

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