BOUNDARY ZEROS OF FUNCTIONS WITH DERIVATIVE IN H^p

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ABSTRACT. It is known that the set of boundary zeros of a function, analytic in the unit disc and with derivative in the Hardy class H^p, is a Carleson set provided $p>1$. In this paper a proof is given which includes the case $p=1$. Peak sets for such functions are investigated and sufficient conditions on subsets of the boundary are given, which guarantee that they are peak sets.

Let D denote the open unit disc $\{z:|z|<1\}$ and let T denote its boundary. For $1\leqq p<\infty$, the Hardy space H^p is the Banach space whose elements are functions f which are analytic in D and have finite norms given by

$$\|f\|_p = \sup_{0\leqq r<1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{it})|^p dt\right\}^{1/p}.$$

The disc algebra A consists of those functions which are analytic in D and have continuous extensions to its closure $\overline{D}$. A function f whose derivative f' is in H^p must belong to A, and this paper is concerned with the boundary zero sets $Z(f)=\{z\in T:f(z)=0\}$ of such functions. Caughran [1] has observed that if $p>1$, then the zero set for such a function is a Carleson set; i.e. a closed subset of T of Lebesgue measure zero whose complementary arcs I_n satisfy $\sum\epsilon_n \log \epsilon_n>-\infty$, where ϵ_n is the length of I_n. In our first theorem, we give a proof of this result which is also valid when $p=1$.

THEOREM 1. *If f is a nonzero analytic function on D with derivative in H^p $(p\geqq 1)$, then its boundary zero set $Z(f)$ is a Carleson set.*

PROOF. If f is a nonzero function in A, $Z(f)$ must be a closed set of Lebesgue measure zero. The complement in T of $Z(f)$ is a disjoint union of open arcs I_n. We assume for simplicity that $-1\in Z(f)$, and let $J_n=(a_n, b_n)$ where e^{ia_n} and e^{ib_n} are the end points of the arc I_n and $-\pi\leqq a_n<b_n\leqq\pi$. Let m denote Lebesgue measure on the line and put $\epsilon_n=m(J_n)$. Let α be a fixed real number with $0<\alpha<1$ and decompose J_n into disjoint sets A_n and B_n, where

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$$A_n = \{t \in J_n : |f(e^{it})| > \epsilon_n^\alpha\}$$

and

$$B_n = \{t \in J_n : |f(e^{it})| \leqq \epsilon_n^\alpha\}.$$

We may assume that $|f|$ is bounded by 1 so that $\log|f|$ is nonpositive. Also $\log|f|$ is integrable since f is a nonzero function in A. These facts together with Jensen's inequality for convex functions give

$$\begin{aligned} -\infty &< \sum \int_{J_n} \log|f(e^{it})|\,dt \\ &\leqq \sum \int_{B_n} \log|f(e^{it})|\,dt \\ &= \sum_{m(B_n)>0} m(B_n) \int_{B_n} \log|f(e^{it})| \frac{dt}{m(B_n)} \\ &\leqq \sum_{m(B_n)>0} m(B_n) \log \int_{B_n} |f(e^{it})| \frac{dt}{m(B_n)} \\ &\leqq \alpha \sum m(B_n) \log \epsilon_n. \end{aligned}$$

Consequently

$$\sum m(B_n) \log \epsilon_n > -\infty. \tag{1}$$

Since $f' \in H^p$ ($p \geqq 1$), the function f, considered as a function on $[-\pi, \pi]$, has bounded variation. For any interval J_n for which $m(A_n) > 0$, the variation of f over J_n exceeds ϵ_n^α because $|f(e^{it})| > \epsilon_n^\alpha$ for $t \in A_n$ and $f(e^{ia_n}) = f(e^{ib_n}) = 0$. Unless $Z(f)$ is finite, we eventually have $0 \geqq \log \epsilon_n \geqq -\epsilon_n^{\alpha-1}$, so there is a constant S such that

$$\begin{aligned} \sum m(A_n) \log \epsilon_n &= \sum_{m(A_n)>0} m(A_n) \log \epsilon_n \\ &\geqq S - \sum_{m(A_n)>0} \epsilon_n^\alpha \\ &\geqq S - \mathrm{Var}\,[f]. \end{aligned}$$

Since f has finite total variation, we have

$$\sum m(A_n) \log \epsilon_n > -\infty. \tag{2}$$

The inequalities (1) and (2) imply that $Z(f)$ is a Carleson set.

The preceding theorem tells us that the set of boundary zeros of a function with derivative in H^1 is necessarily a Carleson set. Conversely, each Carleson set is the set of zeros of such a function; indeed, given a Carleson set E, one can construct a function f with zero set E, such that f and all its derivatives are in A [**3**]. A closely related problem is that of describing the peak sets—or the boundary zero sets of functions with positive real part—for the various above classes of functions. An unpublished result of B. A. Taylor and D. L. Williams shows that a necessary and sufficient condition that $E=Z(f)$ for an f with positive real part and having derivatives of all orders in A, is that E be a finite set. The next theorem implies that a somewhat stronger statement can be made.

THEOREM 2. *Suppose f is analytic in D, $f'\in A$, and* $\operatorname{Re} f>0$ *on D. Then $Z(f)\cap Z(f')=\varnothing$.*

PROOF. Let $u=\operatorname{Re} f$. Then it follows from the Poisson integral formula for u that

$$u(z) \geqq u(0)(1 - |z|)/(1 + |z|) \qquad (z \in D).$$

Hence

$$|f(z)| \geqq u(0)(1 - |z|)/(1 + |z|) \qquad (z \in D).$$

Suppose now that $w\in Z(f)$. Then for $0\leqq r<1$ we have

$$|(f(w) - f(rw))/(w - rw)| = |f(rw)/(1 - r)| \geqq u(0)/2.$$

Thus $|f'(w)|\geqq u(0)/2>0$, so that $w\notin Z(f')$.

COROLLARY. *Under the same hypothesis, $Z(f)$ is a finite set.*

PROOF. If $Z(f)$ has a limit point w, then $f(w)=f'(w)=0$.

If one again demands that $\operatorname{Re} f$ be positive on D, but only requires that f' be in H^p $(1\leqq p<\infty)$, then as we shall show [Theorem 5], it does not necessarily follow that $Z(f)$ is a finite set. Of course, $Z(f)$ must be a Carleson set, and the remainder of this paper is concerned with obtaining some sufficient conditions on a set $E\subset T$ which ensure that E is the zero set for such a function. We begin by stating two results which will be needed for our construction. The proof of the first proceeds along lines almost identical to those which appear in [**3**, Theorem 4.3], while the second is an obvious extension of [**2**, p. 75, exercise 5].

THEOREM 3. *Let $K\subset[-\pi, \pi]$ be a closed set of measure zero, let γ be a number between* 0 *and* 1, *and let ϕ be the extended real-valued function on $[-\pi, \pi]$ defined by $\phi(t)=[\operatorname{dist}(t, K)]^{-\gamma}=[d(t, K)]^{-\gamma}$. If ϕ is integrable and g is the analytic function defined by*

$$(3) \qquad g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\phi(t)dt \qquad (z \in D),$$

then there exists $M>0$ *such that*

$$r\left| g'(re^{i\theta})\right| \leqq M[d(\theta, K)]^{-2}$$

for all $\theta\in[-\pi,\pi]$ *and* $r\in(0,1)$. *Furthermore,* $g'\cdot\exp[-g]$ *is a bounded analytic function.*

THEOREM 4. *Suppose* $1\leqq p<\infty$, $h\in H^p$, *and* F *is an outer function in* H^p. *If the function* h/F *belongs to* $L^p(T)$, *then* h/F *is in* H^p.

THEOREM 5. *Let* $E\subset T$ *be a closed set of measure zero with* $\{I_n\}$ *and* $\{\epsilon_n\}$ *as before. Suppose that* $1\leqq p<\infty$ *and*

$$(4) \qquad \sum \epsilon_n^{1/3p} < \infty.$$

Then there is an analytic function f *such that* $f'\in H^p$, $\operatorname{Re} f>0$ *on* $\bar{D}\sim E$, *and* $E=Z(f)$.

PROOF. Put $K=\{t\in[-\pi,\pi]:e^{it}\in E\}$, $\gamma=1-1/3p$, and define ϕ as in Theorem 3. The convergence condition (4) on the lengths of the complementary components of E is necessary and sufficient that ϕ be integrable. Let g be defined by equation (3) and put $f=1/g$. Then $f\in A$, $\operatorname{Re} f>0$ on $\bar{D}\sim E$, and $E=Z(f)$ (cf. [2, pp. 79–80]). Now $f'=-f^2g'=h/F$ where $h=-f^2g'\exp[-g]$ and $F=\exp[-g]$. Since h is a bounded analytic function (Theorem 3) and F is a bounded outer function, it follows that $f'(e^{it})=\lim_{r\to1}f'(re^{it})$ exists for almost all t and from Theorem 4 that $f'\in H^p$ provided $|f'|^p$ is integrable as a function on T. By Theorem 3, there is a constant M such that if $0<r<1$ and $t\in[-\pi,\pi]$, then

$$\begin{aligned} r\left|f'(re^{it})\right| &= r\left|g'(re^{it})\right|\left|g(re^{it})\right|^{-2} \\ &\leqq M[d(t,K)]^{-2}\left|g(re^{it})\right|^{-2} \\ &\leqq M[d(t,K)]^{-2}[\operatorname{Re} g(re^{it})]^{-2}. \end{aligned}$$

Hence

$$\begin{aligned} \limsup_{r\to1}\left|f'(re^{it})\right| &\leqq M[d(t,K)]^{-2}[\phi(t)]^{-2} \\ &= M[d(t,K)]^{2\gamma-2} \\ &= M[d(t,K)]^{-2/3p}. \end{aligned}$$

Thus $|f'(e^{it})|^p\leqq M^p[d(t,K)]^{-2/3}$ for almost all $t\in[-\pi,\pi]$. But $\gamma=1-1/3p\geqq 2/3$ and $[d(t,K)]^{-\gamma}$ is integrable. It follows that

$[d(t, K)]^{-2/3}$ is integrable and consequently $f' \in H^p$ as required.

REMARK. For a proof that each finite subset of T is a peak set for the class of functions with derivatives of all orders in A, see [**4**, Theorem 3].

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