SOME OSCILLATION PROPERTIES OF SELFADJOINT ELLIPTIC EQUATIONS

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ABSTRACT. In this paper a method is given for generalizing to partial differential equations known nonoscillation theorems for second order ordinary differential equations. As illustrations, two theorems of Hille (one of integral type and one of limit type) are generalized to obtain nonoscillation criteria for second order linear elliptic differential equations on unbounded domains G in n-dimensional Euclidean space R^n.

Let the boundary ∂G of the domain $G \subset R^n$ have a piecewise continuous unit normal. Points in R^n will be denoted by $x = (x_1, x_2, \cdots, x_n)$, and partial differentiation with respect to x_i will be denoted by D_i $(i = 1, 2, \cdots, n)$. We shall consider the elliptic equation

$$(1) \qquad Lu \equiv \sum_{i,j=1}^{n} D_i(a_{ij} D_j u) + bu = 0$$

on G, where the coefficients a_{ij} and b are real and continuous on $G \cup \partial G$ and the matrix (a_{ij}) is symmetric and positive definite on G. The domain of L is defined to be the set of all real-valued continuous functions on $G \cup \partial G$ which have uniformly continuous first partial derivatives in G and for which all derivatives involved in (1) exist and are continuous. A *solution* u of (1) is supposed to belong to the domain of L and satisfy $Lu = 0$ at every point of G. The following notations will be used:

$$G_r = G \cap \{x \in R^n: |x| > r\}, \quad S_r = \{x \in G \cup \partial G: |x| = r\},$$

where $|x|$ is the Euclidean length $(x_1^2 + \cdots + x_n^2)^{1/2}$ of the vector x.

A bounded domain N with $N \cup \partial N \subset G$ is said to be a *nodal domain* of a nontrivial solution u of (1) iff $u = 0$ on ∂N. The differential equation (1) is said to be *oscillatory* in G iff for arbitrary $r > 0$ there exists a solution with a nodal domain in G_r. As Swanson and the author noted in [4], the Clark-Swanson separation theorem [1, Theorem 1,

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p. 888] implies that *every* solution of an oscillatory equation has a zero in G_r for all $r>0$; in other words, every solution of an equation oscillatory in our sense has infinitely many zeros. Equation (1) is said to be *nonoscillatory* in G iff it is not oscillatory in G.

It will be assumed that L is uniformly elliptic in G_s for some $s>0$; i.e., there exists a number $K>0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)z_i z_j \geqq K|z|^2 \tag{2}$$

for all $x\in G_s$, $z\in R^n$.

THEOREM 1. *Let g_1 be the real-valued function defined by*

$$g_1(t) = g_0(t) - K(n-1)(n-3)/(4t^2),$$

with $g_0(r)=\max\{b(x):x\in S_r\}$, $0<r<\infty$. *Then equation* (1) *is nonoscillatory in G if*

$$\limsup_{r\to\infty} r\int_r^\infty g_1^+(t)dt < K/4, \tag{3}$$

where $g_1^+(r)=\max\{g_1(r), 0\}$, $0<r<\infty$.

PROOF. Suppose the contrary, i.e., that (1) is oscillatory in G. Let $p>s$. Then there exists a nontrivial solution u of (1) with a nodal domain N such that $N\cup\partial N\subset G_p$. We now compare (1) with the equation

$$\sum_{i=1}^{n} KD_i^2 v + g_0(r)v = 0, \qquad r = |x|. \tag{4}$$

Because of (2) and the definition of $g_0(r)$, equation (4) majorizes equation (1):

$$\int_N \left\{ \sum (a_{ij} - K\delta_{ij})D_iuD_ju + (g_0 - b)u^2\right\}dx \geqq 0,$$

and it follows by the comparison theorem cited above [1] that *every* solution of (4) has a zero in $N\cup\partial N$, that is, in G_p. However, if the function ρ is a solution of the ordinary differential equation

$$K(r^{n-1}\rho')' + r^{n-1}g_0(r)\rho = 0, \qquad (' = d/dr) \tag{5}$$

then the function v defined by $v(x)=\rho(|x|)$ is a solution of (4). The normal form of (5), obtained by making the oscillation-preserving transformation $\rho=r^{(1-n)/2}w$, is

$$\text{(6)} \qquad Kw'' + [g_0(r) - K(n-1)(n-3)/(4r^2)]w = 0.$$

Since $g_1^+(r)$ is nonnegative, a well-known theorem of Hille [5, Theorem 7, Corollary 1] implies that the equation

$$Ky'' + g_1^+(r)y = 0$$

is nonoscillatory on account of hypothesis (3). Moreover,

$$\text{(7)} \qquad g_1^+(r)/K \geqq g_1(r)/K = g_0(r)/K - (n-1)(n-3)/(4r^2),$$

and it follows from Sturm's comparison theorem [2, p. 208] that equation (6) is nonoscillatory. Thus there exists a solution w of (6) which has no zeros in the interval (r_0, ∞) for some $r_0 > 0$; clearly this solution has no zeros in the interval (c, ∞), where $c > \max\{r_0, s\}$. Hence there exists a solution $v = r^{(1-n)/2}w$ of (4) and a number $c > s$ such that v has no zeros in G_c. This contradiction establishes the theorem.

REMARK. This theorem extends the theorem of Hille cited in the proof to (i) equations with variable leading coefficients, (ii) equations for which the coefficient $b(x)$ is not necessarily eventually positive, (iii) general unbounded domains in n dimensions. A similar remark applies to our next result.

THEOREM 2. *The equation* (1) *is nonoscillatory in* G *if there exists a number* $c > 0$ *such that*

$$\text{(8)} \qquad \int_c^\infty r g_1^+(r)dr < \infty.$$

PROOF. Since we may use the method of Theorem 1, it suffices to show that the equation (6) is nonoscillatory. This is guaranteed by the corresponding criterion of Hille [5, p. 237, Theorem 2] and the Sturm comparison theorem, in view of (7) and (8). The remaining details are omitted.

We shall now give two examples (both of which are modifications of well-known examples for the case $n = 1$) to show that the "sharpness" of the theorems has been preserved in the passage from one to n dimensions. The first example shows that the constant $K/4$ is the best possible in the following sense.

EXAMPLE 1. For every positive integer n there exists an equation, oscillatory in R^n, such that

$$\limsup_{r\to\infty} r\int_r^\infty g_1^+(t)dt = K/4.$$

CONSTRUCTION. Let L be the Schrödinger operator $\nabla^2+b(x)$, $x\in R^n$, and let

$$(9) \qquad b(x) = (n-2)^2/(4r^2) + c/(r\log r)^2, \qquad |x| = r,$$

where c is a number satisfying $c>\frac{1}{4}$. Then it is clear that $K=1$ and

$$\limsup_{r\to\infty} r\int_r^\infty g_1^+(t)dt = \limsup_{r\to\infty} r\int_r^\infty [1/(4t^2)+c/(t\log t)^2]dt = \tfrac{1}{4} = K/4.$$

However, if b is given by (9), equation (1) has particular solutions (namely, solutions depending on r alone) of the form $u=r^{(2-n)/2}(\log r)^s$, where s is a root of the quadratic equation $s^2-s+c=0$. Since $c>\frac{1}{4}$, this quadratic equation has complex roots, and therefore the ordinary differential equation

$$(10) \qquad (r^{n-1}\rho')' + \tfrac{1}{4}r^{n-3}[(n-2)^2 + 4c(\log r)^{-2}]\rho = 0$$

is oscillatory. Let $p>0$ be given. Then there exists a nontrivial solution ρ of (10) with zeros at the points $r=r_1$, r_2, where $r_1>r_2>p$. Hence the function u defined by $u(x)=\rho(r)$, $r=|x|$, is a solution of (1), with a nodal domain $N=\{x\in R^n: r_1<|x|<r_2\}$. Thus there exists a solution u of (1) with a nodal domain N in $\{x\in R^n: |x|>p\}$ for arbitrary $p>0$, since $x\in N\cup\partial N$ implies $|x|\geqq r_1>p$. Hence (1) is oscillatory in R^n.

Our next example shows that the exponent of r in (8) cannot be essentially improved. The example depends on the following n-dimensional form of a one-dimensional result of Glazman [**3**, p. 102]. We state the theorem for equations defined on all of R^n, but it is easily modified, using the methods of an earlier paper of Swanson and the author [**4**], to apply to equations defined on more general (e.g. quasi-conical) domains in R^n.

THEOREM 3. *Let the largest eigenvalue of the matrix* (a_{ij}) *be bounded above in* R^n *by some number* $K_1>0$, *and let* $g(r)=\min\{b(x): |x|=r\}$, $0<r<\infty$. *Then equation* (1) *is oscillatory in* R^n *if*

$$r^2 g(r) \geqq \tfrac{1}{4}K_1(n-2)^2$$

for all sufficiently large r *and*

$$(11) \qquad \limsup_{r\to\infty} (\log r)\int_r^\infty t\{g(t) - K_1(n-2)^2/(4t^2)\}dt = +\infty.$$

PROOF. We compare (1) with the equation

$$\sum_{i=1}^{n} K_1 D_i^2 v + Bv = 0, \qquad B(x) = g(|x|), \quad x \in R^n. \tag{12}$$

By separating the variables we find that (12) has particular solutions of the form $v(x) = \rho(r)$, $0 < r < \infty$, where ρ satisfies the ordinary differential equation

$$K_1(r^{n-1}\rho')' + r^{n-1}g(r)\rho = 0. \tag{13}$$

We reduce (13) to normal form

$$K_1 w'' + \{g(r) - K_1(n-1)(n-3)/(4r^2)\} w = 0, \tag{14}$$

where $\rho(r) = r^{(1-n)/2}w(r)$. The hypothesis (11) implies that the equation (14) is oscillatory in $0 < r < \infty$ by the theorem of Glazman cited above. Thus the equation (13) is also oscillatory, since the transformation $\rho = r^{(1-n)/2}w$ preserves oscillatory behaviour in $0 < r < \infty$. The proof of the theorem may now be completed as in [**4**, Theorem 1]. We omit the details.

EXAMPLE 2. For every positive integer n there exists an equation, oscillatory in R^n, such that, for each $\delta > 0$,

$$\int_s^\infty r^{1-\delta} g_1^+(r) dr < \infty, \qquad (s > 1).$$

CONSTRUCTION. Let L be the Schrödinger operator $\nabla^2 + b(x)$, $x \in R^n$, with b defined by

$$b(x) = (n-2)^2/(4r^2) + 1/(4r^2 \log r), \qquad |x| = r. \tag{15}$$

In this example, $K_1 = 1$, and a routine computation shows that

$$\int_r^\infty t\{g(t) - (n-2)^2/(4t^2)\} dt = +\infty$$

for each $r > 0$. Hence (11) holds and equation (1) is oscillatory if b is given by (15). However, if $s > 1$, then, for each $\delta > 0$,

$$\int_s^\infty r^{1-\delta} g_1^+(r) dr = \int_s^\infty r^{1-\delta}\left[\frac{1}{4r^2} + \frac{1}{4r^2 \log r}\right] dr$$
$$\leq \left(\frac{1}{4} + \frac{1}{4 \log s}\right) \int_s^\infty r^{-1-\delta} dr < \infty.$$

We remark that the equation in this example could have been shown to be oscillatory by appealing to a suitable generalization (along the lines of Theorem 3) of Potter's refinement [**7**] of Leighton's well-known oscillation criterion [**6**].

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