A NOTE ON ISOMORPHISMS OF GROUP ALGEBRAS[1]

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ABSTRACT. In this note, it is shown that, if G_1, G_2 are compact groups, and $C(G_1)$, $C(G_2)$ are the (convolution) algebras of continuous, complex-valued functions on G_1 and G_2 respectively, then the existence of a norm-decreasing algebra-isomorphism of $C(G_1)$ onto $C(G_2)$ ensures that the groups are isomorphic. The corresponding theorem with G_1 and G_2 locally finite discrete groups is also proved.

The main theorem extends the isometric result of Edwards [1] (his result applies to $C_c(G)$ for any locally compact group G) and corresponds to the results for L^1-algebras and measure algebras by Wendel [7], Rigelhof [6], and Greenleaf [2].

This proof is surprisingly longer than the proofs of Wendel and Rigelhof. Wendel's method is not applicable here (since $C(G)$ does not have a *minimal* approximate identity), and we use the result for measure algebras proved by Rigelhof.

We need two results which may be of independent interest. We assume throughout that all compact groups have Haar measure normalized to have total mass one.

LEMMA 1. *For a compact group G with identity* 1, *the linear functional L on $C(G)$ defined by $L(f)=f(1)$ is characterized by the two properties*

(1) $L(f*g)=L(g*f)$ *all* $f,\ g\in C(G)$,

(2) $L(e_\alpha)=n_\alpha^2$ *for each character e_α with degree n_α.*

PROOF. It is sufficient to prove the characterization on the minimal two-sided ideals of $C(G)$. For then, by the Peter-Weyl theorem, the characterization will extend to the whole of $C(G)$.

Let N_α be a minimal two-sided ideal with identity e_α and degree n_α. Then, [4, p. 158], N_α is isomorphic to the full $n_\alpha\times n_\alpha$ matrix algebra over $\mathbf{C}$. If $f\in N_\alpha$ corresponds to the matrix F, then $f(1)=n_\alpha\ \mathrm{trace}(F)$. Since the trace is characterized as the only linear functional satisfying $\mathrm{tr}(AB)=\mathrm{tr}(BA)$ (all matrices A, B) and $\mathrm{tr}(I)=n_\alpha$, (see [3, p. 277]), we have the characterization on N_α.

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COROLLARY. *If G_1 and G_2 are compact groups with identities 1_1 and 1_2 respectively, and $T: C(G_1) \to C(G_2)$ is an algebra-isomorphism, then $(Tf)(1_2) = f(1_1)$ for all $f \in C(G_1)$.*

PROOF. Let e_α be a character on G with degree n_α. Since T is an algebra-isomorphism, Te_α is the identity of a minimal two-sided ideal of $C(G_2)$, and hence is a character on G_2 also with degree n_α. The characterization of Lemma 1 now gives the result.

THEOREM 1. *Let G_1 and G_2 be compact groups, with normalized Haar measure. If $T: C(G_1) \to C(G_2)$ is a norm-decreasing algebra-isomorphism, then T is of the form*

$$(Tf)(x) = \lambda(x) f(\phi x) \qquad (f \in C(G_1),\ x \in G_2)$$

where $\phi: G_2 \to G_1$ is a group isomorphism and a homeomorphism and $\lambda: G_2 \to \boldsymbol{T}$ is a continuous group homomorphism. ($\boldsymbol{T}$ is the circle group $= \{\lambda : |\lambda| = 1\}$.)

PROOF. We show that the dual map $T^*: M(G_2) \to M(G_1)$ is a norm-decreasing isomorphism, and then we use Rigelhof's result in [6]. T^* is clearly norm-decreasing, since T is, so we simply have to show that it is an isomorphism.

We can embed $C(G_i)$ in $M(G_i)$ by

$$\mu_f(g) = (f*g)(1_i) \qquad (f,\ g \in C(G_i)).$$

(In fact, this embedding is an antimonomorphism, since $\mu_f * \mu_g = \mu_{g*f}$.) For f, $g \in C(G_1)$,

$$\begin{aligned} (T^*\mu_{Tf})(g) &= \mu_{Tf}(Tg) \\ &= (Tf*Tg)(1) \qquad \text{(definition)} \\ &= [T(f*g)](1) \\ &= (f*g)(1) \qquad \text{(corollary)} \\ &= \mu_f(g). \qquad \text{(definition)}. \end{aligned}$$

Therefore $T^*\mu_{Tf} = \mu_f$ for all $f \in C(G_1)$, i.e., $T^* = T^{-1}$ on the image of $C(G_2)$ in $M(G_2)$ and hence is an isomorphism. T^* on $M(G_2)$ is an extension of T^{-1} on $C(G_2)$ given by $T^*\mu = \lim_\alpha T^{-1}(\mu * f_\alpha)$ where (f_α) is a central approximate identity in $C(G_2)$. Thus T^* is an isomorphism on $M(G_2)$ and Rigelhof's result applies.

REMARK. The corresponding theorem with G_1 and G_2 locally compact groups, and $T: C_c(G_1) \to C_c(G_2)$ is a norm-decreasing isomorphism is still open. However, it is clear from the proof of the above theorem that it would be sufficient to show that $(Tf)(1_2) = f(1_1)$ for all

$f \in C_c(G_1)$. (But this condition is not necessary. See final remark.)

With a strengthening of Lemma 1 and the Corollary, we can prove Theorem 1 in the case where G_1 and G_2 are discrete locally finite groups. But we first give a generalization of Lemma 1.

LEMMA 2. *Let G be a compact group with identity* 1. *If A is a closed ideal of $C(G)$ with $A*A$ uniformly dense in A, and L is a continuous linear functional on A satisfying*:

(1) $L(f*g)=L(g*f)$ *all* $f, g \in A$,

(2) $L(e_i)=n_i$ *for each character e_i in A with degree n_i, then* $L(f)=f(1)$ (*all* $f \in A$).

PROOF. As in Lemma 1, we know that these properties ensure that $L(f)=f(1)$ for all f in a minimal two-sided ideal N_i of $C(G)$. Since A is the uniform closure of $\sum\{N_i : N_i \subset A\}$ [8, Theorem 5.1], the result follows.

REMARK. It is well known that every closed ideal of A of $C(G)$ satisfies $A*A$ dense in A. The existence of a closed subalgebra of $C(\boldsymbol{T})$ without this property is due to Rider (unpublished, but see [5]).

LEMMA 3. *Let G_1 and G_2 be compact groups. If $T: C(G_1) \to C(G_2)$ is a norm-decreasing monomorphism, then*

(a) *the minimal ideals of* Im T *are minimal ideals of* $C(G_2)$.

(b) $(Tf)(1_2)=f(1_1)$ *all* $f \in C(G_1)$.

PROOF. Let N_α be a minimal two-sided ideal of $C(G_1)$ with identity e_α and dimension n_α^2. Then TN_α is a minimal two-sided ideal of Im T with identity Te_α and dimension n_α^2. In particular TN_α is selfadjoint, and hence, so is Te_α. Now Te_α can be written as a sum of minimal selfadjoint idempotents, say $\sum_{k=1}^n e_k'$. If R_k' is the minimal right ideal of $C(G_2)$ containing e_k' $(k=1 \cdots n)$, then $TN_\alpha \subseteq \sum_{k=1}^n R_k'$. Therefore $\dim(TN_\alpha) \leqq \dim \sum_1^n R_k'$. But

$$\dim \sum_1^n R_k' = \sum_1^n e_k'(1_2) = (Te_\alpha)(1_2) = \|Te_\alpha\| \leqq \|e_\alpha\|$$

$$= \dim(N_\alpha) = \dim(TN_\alpha).$$

That is $\dim(TN_\alpha)=\dim \sum_1^n R_k'$. Therefore TN_α is a right ideal of $C(G_2)$, and, by symmetry, it is also a left ideal. Since it is minimal in Im T, it must be minimal in $C(G_2)$. Therefore $TN_\alpha = N_i'$ for some minimal ideal N_i' in $C(G_2)$, $Te_\alpha = e_i'$ is a character on G_2, and $n_\alpha = n_i$.

For (b), define a linear functional L on $C(G_1)$ by $L(f)=Tf(1_2)$. Then $L(f*g)=L(g*f)$, and, if $Te_\alpha = e_i'$, $L(e_\alpha)=(Te_\alpha)(1_2)=e_i'(1_2)=n_i^2=n_\alpha^2$.

By Lemma 1, $L(f)=f(1_1)$ all $f\in C(G_1)$. Therefore $(Tf)(1_2)=f(1_1)$ for all $f\in C(G_1)$.

REMARK. The condition "norm-decreasing" is essential here. (It was not for the Corollary to Lemma 1.) For, if $\boldsymbol{Z}_2$ is the cyclic group of order 2 generated by z, and $\boldsymbol{D}_3$ is the dihedral group of order 6 generated by x and y, define $T:\boldsymbol{CZ}_2\to\boldsymbol{CD}_3$ by $T(\alpha 1_1+\beta z)=\frac{1}{2}(3\alpha-\beta)(1_2+x)+\beta(y+y^2+xy+xy^2)$, then T is a monomorphism with respect to convolution multiplication (each Haar measure is normalized to have total mass 1), but $T(1_1)=3/2(1_2+x)$.

If T is an isometry in Lemma 3, then Im T, is a closed ideal and this gives us the following extension of Edwards' original result.

THEOREM 2. *If T is an isometric monomorphism of $C(G_1)$ into $C(G_2)$, then T is of the form*

$$(Tf)(x) = \lambda(x)f(\phi x), \qquad f\in C(G_1), \quad x\in G_2,$$

where $\phi:G_2\to G_1$ is a continuous and open group homomorphism, and $\lambda:G_2\to\boldsymbol{T}$ is a continuous group homomorphism.

PROOF. Since T is an isometry, Im T is a closed ideal of $C(G_2)$. Therefore, the adjoint map $T^*:M(G_2)\to M(G_1)$ is norm-decreasing and onto. From Lemma 3, $(Tf)(1_2)=f(1_1)$ for all $f\in C(G_1)$, and the calculation in Theorem 1 gives that $T^*\mu_{Tf}=\mu_f$ for all $f\in C(G_1)$. It follows that, if $T^*(\mu * \nu)=0$ for all $\nu\in L^1(G_2)$, then $T^*\mu=0$. Therefore, by Theorem 3 of [6], T^* and T are of the required form.

REMARK. It is not known whether Theorem 2 is true if T is only assumed to be norm-decreasing.

The converses of Theorems 1 and 2 are true, and can be stated as follows: If ϕ is a continuous and open group homomorphism (resp. isomorphism) of G_2 onto G_1 and $\lambda:G_2\to\boldsymbol{T}$ is a continuous group homomorphism, then, defining T by

$$(Tf)(x) = \lambda(x)f(\phi x), \qquad f\in C(G_1), \quad x\in G_2,$$

it follows that T is an isometric monomorphism (resp. norm-decreasing isomorphism) of $C(G_1)$ into $C(G_2)$.

We can now prove the analogue of Theorem 1 in the case where G_1 and G_2 are locally finite discrete groups.

THEOREM 3. *If G_1 and G_2 are locally finite discrete groups, and $T:C_c(G_1)\to C_c(G_2)$ is a norm-decreasing algebra-isomorphism, then T is of the form $(Tf)(x)=\lambda(x)f(\phi x)$ $(f\in C_c(G_1))$ where $\phi:G_2\to G_1$ is an isomorphism and $\lambda:G_2\to\boldsymbol{T}$ is a homomorphism.*

PROOF. As in Theorem 1, it is sufficient to show that $(Tf)(1_2)=f(1_1)$ all $f\in C_c(G_1)$. For $x\in G_1$, $x\neq 1_1$, let H_1 be the finite subgroup of G_1, generated by x. Define $\delta_x\in C_c(G_1)$ by $\delta_x(y)=0$ $(y\neq x)$, $\delta_x(x)=1$. Then $T\delta_x\in C_c(G_2)$. Let H_2 be the finite subgroup of G_2 generated by the support of $T\delta_x$. Then $T(C(H_1))\subseteq C(H_2)$ and $|H_1|\leqq|H_2|$. Define $T':C(H_1)\to C(H_2)$ by $T'f=(|H_1|/|H_2|)Tf$ each $f\in C(H_1)$ $\subset C_c(G_1)$. Then T' is a norm-decreasing monomorphism of $C(H_1)$ into $C(H_2)$ (normalized Haar measure in H_1 and H_2). By Lemma 3, $(T'f)(1_2)=f(1_1)$ all $f\in C(H_1)$. In particular, $(T'\delta_x)(1_2)=\delta_x(1_1)=0$ since $x\neq 1_1$. Hence $(T\delta_x)(1_2)=0$, and since this is true for all $x\in G_1$, $x\neq 1_1$, $(Tf)(1_2)=f(1_1)$ for all $f\in C_c(G_1)$. This proves the theorem.

Let G_1 and G_2 be locally compact groups and let m_2 be the Haar measure on G_2. If T is any bounded algebra-isomorphism of $C_c(G_1)$ onto $C_c(G_2)$ then, if one replaces m_2 by the Haar measure $(1/\|T\|)m_2$, then $T/\|T\|$ is a norm-decreasing algebra-isomorphism. Thus there are algebra-isomorphisms which are norm-decreasing which do not give rise to group isomorphisms, since there are certainly nonisomorphic groups with isomorphic group algebras.

ADDED IN PROOF. Since submitting this note, the author has found an elementary proof for part of Theorem 2, which is valid for *arbitrary* discrete groups.

Suppose $x\in G_1$, $x\neq 1_1$, and $(T\delta_x)(1_2)=\alpha\neq 0$. Then, if

$$f = (\alpha/|\alpha|)\delta_{1_1} + \delta_x,$$

$\|f\|=1$ and $\|Tf\|>1$, contradicting $\|T\|\leqq 1$. Hence $(T\delta_x)(1_2)=0$ for all $x\in G_1$, $x\neq 1_1$.

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