SOLUTIONS OF $f(x)=f(a)+(RL)\int_a^x(fH+fG)$ FOR RINGS

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ABSTRACT. We show that there is a solution f of the equation

$$f(x) = f(a) + (RL)\int_a^x (fH + fG)$$

such that $f(p)=0$ and $f(q)\neq 0$ for some pair $p, q\in[a, b]$ iff there is a number $t\in[a, b]$ such that one of $1-H(t^-, t)$, $1-H(t, t^+)$, $1+G(t^-, t)$ or $1+G(t, t^+)$ is zero or a right divisor of zero, where f, G and H are functions of bounded variation with ranges in a normed ring N. Furthermore, if N is a field, then for each discontinuity of H on $[a, b]$ there exists $\lambda\in N$ and a finite set of linearly independent nonzero solutions on $[a, b]$ of the equation $f(x)=f(a)+(RL)\int_a^x(fH+fG)\lambda$ such that if f is a solution and has bounded variation on $[a, b]$, then f is a linear combination of this set of solutions. Product integrals are used extensively in the proofs.

1. Definitions and preliminary theorems. For detailed definitions see [**1**, p. 299]. R is the set of real numbers, N is a ring which has a multiplicative identity element 1 and a norm $|\cdot|$ with respect to which N is complete and $|1|=1$; G and H are functions from $R\times R$ to N and functions from R to N are denoted by lower case letters. The symbol $<$ is defined by one of the following statements: (1) if x and $y\in R$, then $x<y$ iff y is less than x, and (2) if x and $y\in R$, then $x<y$ iff x is less than y. The symbols $[x, y]$, $G(x, y)$, $\int_x^y$, ${}_x\prod^y$, etc. imply that $x<y$. $\{x_i\}_0^n$ is a subdivision of $[q, p]$ means $q=x_0<x_1<\cdots<x_n=p$. All sum and product integrals (represented by ${}_a\prod^b G$) are subdivision-refinement-type limits; ${}_x\prod^x G=1$,

$$(RL)\int_a^b (fH + fG) \sim f(y)H(x, y) + f(x)G(x, y)$$

and

$$(m)\int_a^b fH \sim \tfrac{1}{2}[f(x) + f(y)]H(x, y) \quad \text{for } a \leqq x < y \leqq b.$$

$G\in OA^0$ on $[a, b]$ iff $\int_a^b G$ exists and $\int_a^b|G-\int G|=0$; $G\in OM^0$ on $[a, b]$ iff ${}_x\prod^y(1+G)$ exists for $a\leqq x<y\leqq b$ and $\int_a^b|(1+G)-\prod(1+G)|=0$; $G\in OB^0$ on $[a, b]$ iff for a suitably chosen subdivision $\{x_i\}_0^n$ of $[a, b]$ G

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has bounded variation on $[x_{i-1}, x_i]$ for $i=1, 2, \cdots, n$. $G\in OI^0$ on $[x, y]$ means there is a subdivision $\{x_i\}_0^n$ of $[x, y]$ such that if $0<i\leqq n$ then the multiplicative inverse of G exists and is bounded on $[x_{i-1}, x_i]$; appropriate modifications are used for open and half open intervals. $G\in OL^0$ on $[a, b]$ iff $\lim_{x\to p^-} G(x, p)$ and $\lim_{x,y\to p^-} G(x, y)$ exist for $p\in(a, b]$ and $\lim_{x\to p^+} G(p, x)$ and $\lim_{x,y\to p^+} G(x, y)$ exist for $p\in[a, b)$. When confusion is unlikely, phrases such as "on $[a, b]$" will be omitted, $(RL)\int_a^b(fH+fG)$ will be denoted by $\int_a^b fH+fG$, and "the given equation on $[x, y]$" refers to the equation $f(y)=f(x)+(RL)\int_x^y(fH+fG)$.

THEOREM 1. *If H and G are functions from $R\times R$ to N such that $G\in OA^0$ and OB^0 and $H\in OL^0$ on $[a, b]$, then GH and $HG\in OA^0$ and OM^0 on $[a, b]$.*

This is Theorem 2 in [**2**, p. 494].

If H and $G\in OA^0$ and OB^0 and f has bounded variation on $[a, b]$, it follows from Theorem 3.5 [**1**, p. 303] that $f(y)H(x, y)+f(x)G(x, y)\in OA^0$ on $[a, b]$.

THEOREM 2. *If H and G are functions such that H and $G\in OA^0$ and OB^0 and $(1-H)\in OI^0$ on $[a, b]$, then ${}_x\prod^y(1+G)(1-H)^{-1}$ exists for $a\leqq x<y\leqq b$ and, if f is a function, the following statements are equivalent.*

(1) *$f(y)H(x, y)+f(x)G(x, y)\in OA^0$ and $f(x)=f(a)+(RL)\int_a^x(fH+fG)$ for $a\leqq x\leqq b$.*

(2) *If $a\leqq x<y\leqq b$, then $f(y)=f(x)_x\prod^y(1+G)(1-H)^{-1}$.*

PROOF. Let $A=(1+G)(1-H)^{-1}$, then $A-1=(H+G)(1-H)^{-1}\in OB^0$. Since $H\in OB^0$ and $1-H\in OI^0$, then $(1-H)^{-1}\in OL^0$ and, by Theorem 1, $A-1\in OM^0$ and ${}_x\prod^y A$ exists on $[a, b]$. Furthermore, if f is bounded, then $(L)\int_a^b|f(\ )[\prod A-A\ (\ ,\ \ )]|=0$. It follows from Theorem 5.1 [**1**, p. 310] that the two statements are equivalent. Note that ${}_a\prod^x A$ is a bounded function.

2. **Principal theorems.** A corollary to the following theorem is obtained by using the conditions enclosed by brackets in place of those in quotation marks.

THEOREM 3. *If H and $G\in OA^0$ and OB^0 on $[a, b]$, the following statements are equivalent:*

(1) *There is a function f and numbers p and q such that $a\leqq p<q\leqq b$, "$f(p)=0$" [$f(p)\neq 0$], "$f(q)\neq 0$" [$f(g)=0$], f has bounded variation on $[p, q]$ and $f(x)=f(p)+(RL)\int_p^x(fH+fG)$ for $x\in[p, q]$.*

(2) *There is a number $t\in[a, b]$ such that $t\neq a$ and "$1-H(t^-, t)$" [$1+G(t^-, t)$] is zero or a right divisor of zero or such that $t\neq b$ and "$1-H(t, t^+)$" [$1+G(t, t^+)$] is zero or a right divisor of zero.*

PROOF. (1)→(2). Let S be the number set such that $x\in S$ iff $x\in[p, q]$ and $f(x)=0$; then S has a least upper bound t and $p\leqq t\leqq q$. If $f(t)\neq 0$, then $a\leqq p<t$,

$$f(t) = f(p) + \int_p^t fH + fG = f(t^-) + f(t)H(t^-, t)$$

and

$$0 = f(t^-) = f(t)[1 - H(t^-, t)]$$

and therefore $1-H(t^-, t)$ is zero or a right divisor of zero.

If $f(t)=0$, then $t<q\leqq b$ and

$$f(t^+) = f(p) + \int_p^{t^+} fH + fG = f(t) + \int_t^{t^+} fH + fG = f(t^+)H(t, t^+)$$

and therefore $f(t^+)[1-H(t, t^+)]=0$ and $1-H(t, t^+)$ is zero or a right divisor of zero, provided $f(t^+)\neq 0$. Suppose $f(t^+)=0$; then there is a number c such that $t<c<q$ and $(1-H)\in OI^0$ on $(t, c]$; hence, by Theorem 2, if $x\in(t, c]$, then

$$f(x) = f(t^+) + \int_{t^+}^x fH + fG = f(t^+)\,{}_{t^+}\prod^x (1 + G)(1 - H)^{-1} = 0.$$

Therefore, t is not the least upper bound of S.

(2)→(1). Suppose $t\neq a$ and $1-H(t^-, t)$ is zero or a right divisor of zero; let $p=a$, $q=t$, k be a nonzero element of N such that $k[1-H(t^-, t)]=0$, and let f be the function such that $f(x)=0$ for $x\in[a, t)$ and $f(t)=k$. If $x\in[a, q)$, then $f(a)+\int_a^x fH+fG=0=f(x)$. If $x=q=t$, then

$$f(a) + \int_a^t fH + fG = f(t^-) + \int_{t-}^t fH + fG = f(t)H(t^-, t)$$
$$= f(t) - k[1 - H(t^-, t)] = f(t) = f(x).$$

Suppose $t\neq b$ and $1-H(t, t^+)$ is zero or a right divisor of zero. There is a number q such that $t<q<b$ and such that $1-H\in OI^0$ on $(t, q]$. Also, there is a nonzero element $k\in N$ such that $k[1-H(t, t^+)]=0$. Let $p=a$ and define f to be the function such that $f(x)=0$ for $x\in[a, t]$ and

$$f(x) = k_{t^+}\prod^x (1 + G)(1 - H)^{-1} \quad \text{for } x \in (t, q];$$

then $f(t^+)=k$. If $x\in(t, q]$, then

$$f(p) + \int_p^x fH + fG$$
$$= \left(\int_t^{t^+} + \int_{t+}^x\right)(fH + fG) = f(t^+)H(t, t^+) + \int_{t+}^x fH + fG$$
$$= k - k[1 - H(t, t^+)] + \int_{t+}^x fH + fG = k_{t+}\prod^x (1 + G)(1 - H)^{-1}$$
$$= f(x).$$

Since H and $G \in OB^0$ and f is bounded on $[p, q]$, then $\int fH + fG$ and f have bounded variation on $[p, q]$.

PROOF OF COROLLARY. Since $f(y) = f(x) + (RL)\int_x^y fH + fG$ for $a \leqq x < y \leqq b$ iff for $a \leqq x < y \leqq b$

$$f(x) = f(y) - (RL)\int_x^y fH + fG = f(y) + (RL)\int_y^x f(-g) + f(-h),$$

where $g(y, x) = G(x, y)$ and $h(y, x) = H(x, y)$, it follows that this corollary is a special case of the preceding theorem with $-g$ and $-h$ playing the roles of H and G, respectively.

LEMMA. *If* $f(x) = f(a) + (RL)\int_a^x (fH + fG)$ *for* $x \in [a, b]$, *then*
(1) *if* $x \in (a, b]$, $f(x)[1 - H(x^-, x)] = f(x^-)[1 + G(x^-, x)]$, *and*
(2) *if* $x \in [a, b)$, $f(x^+)[1 - H(x, x^+)] = f(x)[1 + G(x, x^+)]$.

THEOREM 4. *Given*: $a \leqq p \leqq b$; *H and $G \in OA^0$ and OB^0 on* $[a, b]$; *if* $a < p$, *then* $H(p^-, p) = 1$; *if* $p < b$, *then* $H(p, p^+) = 1$ *and* $1 + G(p, p^+)$ *is not a right divisor of zero*; *there is a function f of bounded variation on* $[a, b]$ *such that* $f(p) \neq 0$ *and* $f(x) = f(a) + (RL)\int_a^x (fH + fG)$ *for* $x \in [a, b]$; *and u is a function such that* $u(x) = 0$ *if* $x \neq p$.
Conclusion. If $x \in [a, b]$, *then* $u(x) = u(a) + (RL)\int_a^x (uH + uG)$.

PROOF. If $a < p \leqq b$, then $u(x) = 0$ for $a \leqq x < p$ and

$$u(a) + \int_a^p uH + uG = u(p)H(p^-, p) = u(p).$$

If $a \leqq p < b$, then it follows from the lemma that

$$f(p)[1 + G(p, p^+)] = f(p^+)[1 - H(p, p^+)] = 0;$$

hence, $1 + G(p, p^+) = 0$ and, if $x \in (p, b]$, then

$$u(p) + \int_p^x uH + uG = u(p) + u(p^+)H(p, p^+) + u(p)G(p, p^+)$$
$$= u(p)[1 + G(p, p^+)] = 0 = u(x).$$

If $a<p<b$, it follows from the two preceding results that $u(x)=u(a)+(RL)\int_a^x(uH+uG)$ for $x\in[a, b]$.

In the following theorems the symbol $\langle p, q\rangle$ denotes a subset of $[a, b]$ such that

(1) $1-H\in OI^0$ on $\langle p, q\rangle$ and $(p, q)\subseteq\langle p, q\rangle\subseteq[p, q]$;

(2) H has a discontinuity of 1 at p provided $p\neq a$, and at q provided $q\subseteq b$; and

(3) $p\in\langle p, q\rangle$ iff $H(p, p^+)\neq 1$, and $q\in\langle p, q\rangle$ iff $H(q^-, q)\neq 1$. Also, if $p\in\langle p, q\rangle$ and $a\neq p$, then p', p^* denotes p^-, p; if $p\notin\langle p, q\rangle$, then p', p^* denotes p, p^+; if $q\notin\langle p, q\rangle$, then q', q^* denotes q^-, q; and if $q\in\langle p, q\rangle$, then q', q^* denotes q, q^+; if $a\in\langle p, q\rangle$, the $p^*=a$.

THEOREM 5. *Given.* $a\leqq p<q\leqq b$; *H and* $G\in OA^0$ *and* OB^0; $(1-H)\in OI^0$ *on* $\langle p, q\rangle$; *either* $a\in\langle p, q\rangle$, *or* $a<p\in\langle p, q\rangle$ *and* $H(p^-, p)=1$, *or* $a\leqq p\notin\langle p, q\rangle$ *and* $H(p, p^+)=1$; *either* $b\in\langle p, q\rangle$, *or* $b\geqq q\notin\langle p, q\rangle$ *and* $H(q^-, q)=1$, *or* $b>q\in\langle p, q\rangle$ *and* $H(q, q^+)=1$; *if* $x\in[a, b]$, *then neither of* $1+G(x^-, x)$ *or* $1+G(x, x^+)$ *is a right divisor of zero; there is a function f with bounded variation on* $[a, b]$ *and a number* $t\in\langle p, q\rangle$ *such that* $f(t)\neq 0$ *and* $f(x)=f(a)+(RL)\int_a^x(fH+fG)$ *for* $x\in[a, b]$; *u is a function such that* $u(x)=0$ *for* $x\notin\langle p, q\rangle$, $u(p^*)=1$, *and if* $x\in\langle p, q\rangle$ *then* $u(x)={}_{p^*}\prod^x(1+G)(1-H)^{-1}$.

Conclusion. (1) *If* $x\in[a, b]$, *then* $u(x)=u(a)+(RL)\int_a^x(uH+uG)$.

(2) *If the function w has bounded variation and is a solution of the given equation on* $[a, b]$, *then* $w(x)=w(p^*)u(x)$ *for* $x\in\langle p, q\rangle$ *and, if there exists a number* $c\in\langle p, q\rangle$ *such that* $w(c)\neq 0$, *then* $w(p^*)\neq 0$.

PROOF OF (1). If $a\notin\langle p, q\rangle$, it follows from the preceding theorem that u is a solution on $[a, p^*]$; hence, if $x\in\langle p, q\rangle$, then

$$\begin{aligned} u(a)+\int_a^x uH+uG &= u(p^*)+\int_{p^*}^x uH+uG \\ &= u(p^*)_{p^*}\prod^x(1+G)(1-H)^{-1}=u(x). \end{aligned}$$

Suppose $b\notin\langle p, q\rangle$. It follows from Theorem 2 that u is a solution on $\langle p, q\rangle$; hence, if $x\in[q^*, b]$, then $u(q^*)=0$ and

$$\begin{aligned} u(p^*)+\int_{p^*}^x uH+uG &= u(q')+\left(\int_{q'}^{q^*}+\int_{q^*}^x\right)(uH+uG) \\ &= u(q')+u(q^*)H(q', q^*)+u(q')G(q', q^*) \\ &= u(q')[1+G(q', q^*)]=0=u(x), \end{aligned}$$

provided one of $u(q')$ or $1+G(q', q^*)$ is zero.

In order to show that the preceding requirement is satisfied, we will consider two cases: $f(q')\neq 0$ and $f(q')=0$. If $f(q')\neq 0$, then

$$0 = f(q^*)[1 - H(q', q^*)] = f(q')[1 + G(q', q^*)]$$

and $1+G(q', q^*)=0$ because $1+G(q', q^*)$ is not a right divisor of zero. If $f(q')=0$, then it follows from the corollary to Theorem 3 that there is a number z such that $t<z\leqq q'$ and such that $1+G(z', z^*)=0$; hence, $u(q')={}_{p^*}\prod^{q'}(1+G)(1-H)^{-1}=0$.

If $a\notin\langle p, q\rangle$ and $b\notin\langle p, q\rangle$, it follows from the two preceding results that

$$u(x) = u(a) + (RL)\int_a^x (uH + uG) \quad \text{for } x \in [a, b].$$

Let c be a number such that $p<c<q$; then, if $q^*\leqq x\leqq b$, it follows that

$$u(a) + \int_a^x uH + uG = u(c) + \int_c^x uH + uG = u(x).$$

Proof of (2). If $x\in\langle p, q\rangle$, then, by Theorem 2,

$$w(x) = w(a) + \int_a^x wH + wG = w(p^*) + \int_{p^*}^x wH + wG$$
$$= w(p^*)_{p^*}\prod^x (1 + G)(1 - H)^{-1} = w(p^*)u(x)$$

and $w(x)=0$ if $w(p^*)=0$. Hence, if $c\in\langle p, q\rangle$ and $w(c)\neq 0$, then $w(p^*)\neq 0$.

Theorem 6. *Given. H and G are functions from $R\times R$ to N such that H and $G\in OA^0$ and OB^0 on $[a, b]$; if $x\in(a, b]$ and $1-H(x^-, x)^{-1}$ does not exist, then $H(x^-, x)=1$; if $x\in[a, b)$ and $[1-H(x, x^+)]^{-1}$ does not exist, then $H(x, x^+)=1$; if $x\in[a, b]$, then neither of $1+G(x^-, x)$ or $1+G(x, x^+)$ is a right divisor of zero.*

Conclusion. There is a finite set of linearly independent solutions of the equation $f(x)=f(a)+(RL)\int_a^x (fH+fG)$ on $[a, b]$ such that a function f is a linear combination of this set iff f has bounded variation on $[a, b]$ and f is a solution to the given equation on $[a, b]$.

Proof. It is assumed that the equation has at least one nonzero solution. Let $\{x_i\}_0^n$ be the subdivision of $[a, b]$ such that $x\in\{x_i\}_1^{n-1}$ iff $H(x^-, x)=1$ or $H(x, x^+)=1$; then $(1-H)\in OI^0$ on $\langle x_{i-1}, x_i\rangle$ for $i=1, 2, \cdots, n$.

Let P be the set of integers such that $i\in P$ iff $0<i\leqq n$ and there is a solution f on $[a, b]$ and a number $t\in\langle x_{i-1}, x_i\rangle$ such that $f(t)\neq 0$. For

each $i\in P$, define $c_i=x_{i-1}^*$ and u_i to be the function defined in Theorem 5, where c_i corresponds to p^* and $u_i(c_i)=1$.

Let O be the set of integers such that $i\in Q$ iff $0\leqq i\leqq n$ and $x_i \notin \bigcup_{i\in P}\langle x_{i-1}, x_i\rangle$ and there is a solution f on $[a, b]$ such that $f(x_i)\neq 0$. For $i\in Q$, define w_i to be the function such that $w_i(x_i)=1$ and $w_i(x)=0$ if $x\neq x_i$; it follows from Theorem 4 that w_i is a solution of the equation on $[a, b]$.

The set $\{u_i\}_{i\in P}\cup\{w_i\}_{i\in Q}$ of functions is the desired set. Since each function belonging to the set has bounded variation and is a solution of the equation on $[a, b]$, then each linear combination of these functions is a solution and has bounded variation on $[a, b]$. If $\{k_i\}_{i\in P}$ and $\{h_i\}_{i\in Q}$ are subsets of N and if $m\in P$, then

$$\sum_{i\in P} k_i u_i(c_m) + \sum_{i\in Q} h_i w_i(c_m) = k_m u_m(c_m) = k_m;$$

a similar result holds for $m\in Q$. Therefore, if

$$\sum_{i\in P} k_i u_i(x) + \sum_{i\in Q} h_i w_i(x) = 0$$

for all $x\in[a, b]$, then each of the coefficients is zero; hence, the functions are linearly independent.

If f is a solution of the equation and has bounded variation on $[a, b]$ and $x\in[a, b]$, then the summation

$$\sum_{i\in P} f(c_i)u_i(x) + \sum_{i\in Q} f(x_i)w_i(x) = g(x)$$

simplifies as follows:

(1) if $i\in P$ and $x\in\langle x_{i-1}, x_i\rangle$, then $g(x)=f(c_i)u_i(x)=f(x)$, by Theorem 5;

(2) if $i\in Q$ and $x=x_i$, then $g(x)=f(x_i)w_i(x)=f(x_i)=f(x)$; and

(3) if $x\in\langle x_{i-1}, x_i\rangle$ and $i\notin P$ or if $x=x_i$ and $i\notin Q$, then $g(x)=0$.

From the definition of P and Q, if the conditions in (3) are satisfied, then $f(x)=0$. Hence, the above summation is $f(x)$ for $x\in[a, b]$.

3. **Comments.** If N is a field and H and $G\in OA^0$ and OB^0 on $[a, b]$, then each of the equations

$$f(x) = (RL)\int_a^x (fH + fG)\lambda,\quad f(x) = (R)\int_a^x fH\lambda$$

and

$$f(x) = (m)\int_a^x fH\lambda$$

has a solution on $[a, b]$ iff H has a discontinuity k on $[a, b]$, in which case $\lambda = k^{-1}$. If k is such a discontinuity of H, then there is a largest number $p \in [a, b]$ at which the discontinuity occurs and the function f can be defined on $[a, b]$ as in Theorem 3(2)→(1). The set of λ's may be infinite but cannot be uncountable. The possibility that the equation $f(x) = f(a) + (RL)\int_a^x (fH + fG)$ has a solution f on $[a, b]$ for which $f(a) \neq 0$ depends on the order of occurrence and relative values of the discontinuities of H and G.

The following conjectures are probably true.

1. Similar theorems will hold for the equations

$$f(x) = f(a) + (RL)\int_a^x (Hf + Gf),$$

$$f(x) = f(a) + (RL)\int_a^x (fH + Gf)$$

and

$$f(x) = f(a) + (RL)\int_a^x (Hf + fG).$$

2. The set R can be any linearly ordered set [**4**, p. 149].

3. In Theorems 4 and 5 the restrictions on $1 - H$ can be relaxed to permit $1 - H(x^-, x)$ and $1 - H(x, x^+)$ to be right divisors of zero.

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