PROPERTIES Γ AND L FOR TYPE II_1 FACTORS

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ABSTRACT. Using the new concept of central sequences introduced by Dixmier and Lance, it is proved that for a type II_1 factor on a separable Hilbert space properties Γ and L are equivalent.

Let $\mathfrak{A}$ be a countably generated type II_1 factor. (The referee has pointed out that this is equivalent to the existence of a faithful scalar valued trace on $\mathfrak{A}$.) For $A \in \mathfrak{A}$ let $\|A\|$ denote the operator bound of A and let $|A| = [\mathrm{tr}(A^*A)]^{1/2}$ where $\mathrm{tr}(A)$ is the normalized trace on $\mathfrak{A}$.

Using results of Dixmier in [1] (stated for $\mathfrak{A}$ acting in a separable space but clearly valid so long as $\mathfrak{A}$ is countably generated) we show that for $\mathfrak{A}$ properties L of Pukanszky [5] and Γ of Murray and von Neumann [4] are equivalent.

We recall the following from [3].

DEFINITION 1. Let (A_i) be a bounded (i.e. $\|A_i\| \leqq M$) sequence in $\mathfrak{A}$. Let $[A, B] = AB - BA$. We say that (A_i) is a *central sequence* iff for every $A \in \mathfrak{A}$ $|[A, A_i]| \to 0$ as $i \to \infty$. We say that (A_i) is *trivial* iff there is a bounded sequence of scalars (λ_i) such that $|A_i - \lambda_i| \to 0$ as $i \to \infty$.

Since $\mathfrak{A}$ is countably generated, the following definition of property Γ is identical with that in [4].

DEFINITION 2. $\mathfrak{A}$ has property Γ (respectively property L) iff there is a central sequence (U_i) in $\mathfrak{A}$ such that the U_i are unitary and such that $\mathrm{tr}(U_i) = 0$ for each U_i (respectively U_i unitary and $U_i \to 0$ weakly).

THEOREM 3. *$\mathfrak{A}$ has property Γ iff $\mathfrak{A}$ has property L.*

PROOF. Dixmier proved [1, Proposition 1.10] that $\mathfrak{A}$ has Γ iff $\mathfrak{A}$ has some nontrivial central sequence. Suppose that $\mathfrak{A}$ has L, and that (U_i) is the relevant sequence. Then $U_i \to 0$ weakly. We assert that (U_i) is nontrivial and hence that $\mathfrak{A}$ has Γ. Indeed, suppose that (λ_i) is a bounded sequence of scalars such that $|U_i - \lambda_i| \to 0$. Then $U_i - \lambda_i \to 0$ strongly. Without loss of generality we can assume that $\lambda_i \to \lambda$. Hence $U_i \to \lambda$ strongly. But by assumption $U_i \to 0$ weakly.

Presented to the Society, March 27, 1970; received by the editors July 15, 1969.
AMS Subject Classifications. Primary 4665.
Key Words and Phrases. Factor, type II_1, property Γ, property L, central sequence.

Hence $\lambda = 0$ and the sequence $U_i \to 0$ strongly, contradicting the unitarity of the U_i.

Suppose conversely that $\mathfrak{A}$ has Γ and that (U_i) is the relevant sequence. For every $A \in \mathfrak{A}$, we have $U_n A - A U_n \to 0$ strongly. By weak compactness of the unit ball of operators we may assume that $U_i \to V$ weakly. It follows that $VA = AV$ for all $A \in \mathfrak{A}$, so that $V = \lambda I$. However, $\mathrm{tr}(U_i) = 0$ for all U_i, whence $\lambda = 0$, i.e. $\mathfrak{A}$ has property L. Q.E.D.

COROLLARY 4. *Let $\mathfrak{A}$ have Γ or L. Then there is a sequence (U_i) demonstrating Γ and L simultaneously.*

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