A DUALITY BETWEEN SPHERES AND SPHERES WITH ARCS

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ABSTRACT. The purpose of this paper is to extend the authors duality between nearly tame spheres and nearly tame arcs to a duality between wild spheres and tame spheres with nearly tame arcs properly attached, one arc for each wild point of the wild sphere.

1. Let S^3 be the one point compactification of Euclidean 3-space R^3 and let S^2 and B^3 be the unit 2-sphere and 3-ball in R^3 centered at the origin, respectively. For $X \subset S^3$, let Cl X denote the closure of X and $V_\epsilon(X)$ denote the ϵ-neighborhood of X.

A set X in S^3 is *locally flat* at a point $x \in X$ if there is a neighborhood U of x and a canonical set Y in R^3 such that $(U, U \cap X)$ is topologically equivalent to (R^3, Y).

2. Let $\mathcal{S}$ be the collection of all 2-spheres in S^3. Let $\Sigma \in \mathcal{S}$ and let C_0 and C_1 be the closed complementary domains of Σ in S^3. By [4] or [5] there are imbeddings $h_i: C_i \to S^3$ such that $\mathrm{Cl}(S^3 - (h_0(C_0) \cup h_1(C_1)))$ is an annulus. Hence there is an imbedding

$$g: S^2 \times [0, 1] \to \mathrm{Cl}\ (S^3 - (h_0(C_0) \cup h_1(C_1)))$$

such that $g(S^2 \times i) = h_i(\Sigma)$ and $h_0^{-1}g(y, 0) = h_1^{-1}g(y, 1)$ for each $y \in S^2$. Without loss of generality $g(y, \frac{1}{2}) = y$.

Let X_i be the set of points x in S^2 such that $h_i(\Sigma)$ is not locally flat at $g(x, i)$, $i = 0, 1$, and let $X = X_0 \cup X_1$. Let $a_x = g(x \times [0, \frac{1}{2}])$ if $x \in X_0 - X_1$, let $a_x = g(x \times [\frac{1}{2}, 1])$ if $x \in X_1 - X_0$ and let $a_x = g(x \times [0, 1])$ if $x \in X_0 \cap X_1$. It follows from Lemma 6 of [6] that each arc a_x is cellular. Then by Lemma 4 of [6] or Theorem 10 of [7] each arc is locally flat except possibly at one end point.

Let $S = S^2$. Then $S \cup (\bigcup a_x)$ satisfies the following properties:

(1) Each arc a_x intersects the flat (tame) 2-sphere S at one point x,

(2) Each a_x is locally flat (locally tame) except possibly at one end point not on S,

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(3) $S \cup (\bigcup a_x)$ is not locally flat at the end points of the arcs which are not on S but is locally flat at all other points,

(4) If a_y is locally flat at an end point p_y not on S, then p_y is a limit point of end points of other arcs in the collection $\{a_x\}$,

(5) $\bigcup a_x$ is closed,

(6) $\{a_x\}$ is the set of nondegenerate elements of an upper semicontinuous decomposition $d: S^3 \to d(S^3)$ such that $d(S^3)$ is S^3.

A *sphere with arcs* is a set $S \cup (\bigcup a_x)$ which satisfies properties (1)–(5). Let $\mathcal{A}$ be the collection of spheres with arcs each of which also satisfies property (6). Then we can define a correspondence $\Gamma: \mathcal{S} \to \mathcal{A}$ by $\Gamma(\Sigma) = S \cup (\bigcup a_x)$ as indicated in the preceding paragraphs, although Γ is not a well-defined function. However, if $\mathcal{S}_*$ and $\mathcal{A}_*$ are the collections of equivalence classes of $\mathcal{S}$ and $\mathcal{A}$, respectively, then it is easy to see that Γ induces a function $\Gamma_*: \mathcal{S}_* \to \mathcal{A}_*$.

Now define $\Psi: \mathcal{A} \to \mathcal{S}$ by $\Psi(S \cup (\bigcup a_x)) = d(S)$, where d is the decomposition map of property (6). It is easy to see that Ψ induces a function $\Psi_*: \mathcal{A}_* \to \mathcal{S}_*$.

THEOREM 1. *Γ_* and Ψ_* are inverses of each other.*

This theorem (to be proved in §4) establishes the duality between $\mathcal{S}$ and $\mathcal{A}$. A dual of $\Sigma \in \Sigma$ is any $S \cup (\bigcup a_x) \in \Gamma_*(\mathcal{S}) \subset \mathcal{A}$ and a dual of $S \cup (\bigcup a_x) \in \mathcal{A}$ is any $\Sigma \in \Psi_*(S \cup (\bigcup a_x)) \subset \mathcal{S}$.

Notice that a collection of arcs alone does not determine a unique sphere. In particular, Example 1.2 of Fox and Artin [**3**] and its reflection can be attached to a tame 2-sphere so that the dual wild sphere may have both or just one of its complementary domains simply connected.

Properties (3) and (6) are necessary in order to keep $\mathcal{A}$ small enough to establish the duality; it is easy to find an example to show that property (3) is necessary and Ball [**1**] has described a sphere with arcs which fails to satisfy property (6). Properties (4) and (5) insure that an arc a_x is in the collection of arcs if and only if x corresponds to a wild point of the dual sphere.

3. In this section we will use polar coordinates, that is, $p = (\theta, r)$ where r is the distance from p to the origin and θ is the projection (from the origin) of p onto S^2.

THEOREM 2. *If $S \cup (\bigcup a_x)$ is a sphere with arcs such that $S = S^2$ and $\bigcup a_x \subset B^3$, then S^2 has a collar C in B^3 such that $a_x \cap C$ is the collar at x. In fact, there is a homeomorphism $h: S^2 \times [0, 2] \twoheadrightarrow B^3$ such that $h(x \times [1, 2] \cup a_x) = a_x$ for each $x \in X$.*

PROOF. Since $S^2\cup(\bigcup a_x)$ is locally flat, there is a finite open (in B^3) cover $\{C_1, \cdots, C_n\}$ of S^2 by local collars in B^3 such that, for $x\in C_i\cap X$, $a_x\cap C_i$ is the collar at x. Let $U_i=C_i\cup((C_i\cap S^2)\times[1, 2])$ and let $\{V_1, \cdots, V_n\}$ be an open (in S^2) cover of S^2 such that $\mathrm{Cl}\ V_i\subset U_i$. Let h_i be a homeomorphism of $S^2\times[0, 2]$ onto itself such that $h_i|(S^2\times[0, 2])-U_i=1$, $h_i(V_i)\subset\mathrm{Int}\ B^3$ and, for $x\in C_i\cap X$,

$$h_i((x\times[1, 2])\cup a_x) = (x\times[1, 2])\cup a_x.$$

Let $h=h_n\cdots h_1$ and let g be a homeomorphism of $S^2\times[0, 2]$ onto B^3 such that $g(x, 2)=(x, 1)$ for $x\in S^2$ and such that $gh((x\times[1, 2])\cup a_x)=a_x$ for $x\in X$. Then gh is the required homeomorphism.

THEOREM 3. *If $S\cup(\bigcup a_x)$ is a sphere with arcs and $\bigcup a_x$ is contained in one of the closed complementary domains B of S, then $S\cup(\bigcup a_x)$ satisfies property* (6). *In fact, if $S=S^2$ and $B=B^3$, then there is a map $g:S^2\times[0, 2]$ onto itself such that g maps each a_x to a point, $g(x\times[1, 2])=(x\times[1, 2])\cup a_x$, for $x\in X$, and $g|S^2\times 2=1$.*

PROOF. Without loss of generality $S=S^2$ and $\bigcup a_x\subset B^3$. Let p_x be the end point of a_x not on S^2. By techniques similar to those in the proof of Theorem 2, we can prove that given a sequence of positive numbers $\{\epsilon_i\}$ there is a sequence of homeomorphisms $\{h_i\}$ of S^3 onto itself such that

$$h_i|S^3 - V_{\epsilon_i}(\bigcup a_x) = h_{i-1},\quad h_i((x\times[1,2])\cup a_x) = (x\times[1,2])\cup a_x$$

and

$$h_i(\bigcup a_x)\subset V_{1/i}(\bigcup p_x).$$

By a proper choice of the ϵ_i, these homeomorphisms converge to a map h of S^3 onto itself such that $h|S^3-\bigcup a_x$ is a homeomorphism, $h(a_x)=p_x$, $h(x\times[1, 2])=(x\times[1, 2])\cup a_x$ and $h|S^3-S^2\times[0, 2]=1$. Then $g=h|S^2\times[0, 2]$ is the required map.

4. **Proof of Theorem** 1. Let $\Sigma\in\mathcal{S}$. We use the notation of §2; hence, $\Gamma(\Sigma)=S\cup(\bigcup a_x)$ and $\Psi(S\cup(\bigcup a_x))=d(S)$. Let k be a map of $d(S^3)$ onto itself such that $k(dg(S^2\times[0, 1]))=d(S)$ and $k|d(S^3)-dg(S^2\times[0, 1])$ is a homeomorphism onto $d(S^3)-d(S^2)$. This is possible since a manifold with boundary plus a closed collar is canonically homeomorphic to the manifold with boundary. Define $f:S^3\twoheadrightarrow d(S^3)$ by

$$\begin{aligned} f(x) &= kdh_0(x), & x&\in C_0,\\ &= kdh_1(x), & x&\in C_1. \end{aligned}$$

Evidently f is a homeomorphism and $f(\Sigma)=d(S)$. Thus Σ is equivalent to $d(S)=\Psi\Gamma(\Sigma)$ and $\Psi_*\Gamma_*=1$.

Let $S\cup(\bigcup a_x)\in\mathcal{A}$. By Theorem 2 there is a homeomorphism $h:S\times[-1,2]\to S^3$ such that $h(y,\frac{1}{2})=y$ for each $y\in S$ and $h(x\times[-1,2])\supset a_x\cap h(S\times[-1,2])$ for each $x\in X$. Let f_1 be a map of S^3 onto itself such that

$$f_1|S^3-h(S\times[-1,2])=1,\quad f_1(h(y\times[1,2]))=h(y\times[-1,2])$$

and

$$f_1(h(y\times[0,1]))=h(y,\tfrac{1}{2})$$

for each $y\in S$. By Theorem 3 there is a map f_2 of S^3 onto itself such that f_2 maps each component of $a_x-h(x\times[0,1])$ to a point for each $x\in X$, f_2 is a homeomorphism elsewhere, $f_2(a_x)=a_x$ and $f_2|S=1$. It follows from property (6) that there is a map f_3 of S^3 onto itself such that $f_3(f_2(h(y\times[0,1])))$ is a point for all $y\in S$ and f_3 is a homeomorphism elsewhere. Then $f=f_3f_2f_1^{-1}$ is a map of S^3 onto itself such that $f(a_x)$ is a point for each $x\in X$ and $f|S^3-\bigcup a_x$ is a homeomorphism. Then $\Sigma=f(S)\in\Psi_*(S\cup(\bigcup a_x))$.

Let C_0 and C_1 be the closed complementary domains of Σ in S^3. Define an imbedding $h_i:C_i\to S^3$ such that $f_3h_i:C_i\to C_i$ is the identity. Define an imbedding $g:S\times[0,1]\to S^3$ by $g(y,t)=f_2h(y,t)$. Then $g(S\times i)=h_i(\Sigma)$,

$$\overset{-1}{h_0}g(y,0)=f_3f_2h(y,0)=f_3f_2h(y,1)=\overset{-1}{h_1}g(y,1)$$

and $g(y,\frac{1}{2})=f_2h(y,\frac{1}{2})=f_2(y)=y$. From this it follows that $S\cup(\bigcup a_x)\in\Gamma_*(\Sigma)$ and hence $\Gamma_*\Psi_*=1$.

5. Let $\mathcal{S}_1$ be the collection of 2-spheres in S^3 each of which is locally flat except at a countable number of points and let $\mathcal{A}_1$ be the collection of spheres with arcs in S^3 such that the collection of arcs is countable. By Theorem 1 of [2] each set in $\mathcal{A}_1$ satisfies property (6).

THEOREM 4. *The duality between $\mathcal{S}$ and $\mathcal{A}$ restricts to a duality between $\mathcal{S}_1$ and $\mathcal{A}_1$.*

Let $\mathcal{S}_2$ be the collection of 2-spheres in S^3 each of which has a closed complementary domain which is a closed 3-cell and let $\mathcal{A}_2$ be the collection of spheres with arcs in S^3 such that all the arcs are contained in one of the closed complementary domains of the tame sphere. By Theorem 3 each set in $\mathcal{A}_2$ satisfies property (6).

THEOREM 5. *The duality between* $\mathcal{S}$ *and* $\mathcal{A}$ *restricts to a duality between* $\mathcal{S}_2$ *and* $\mathcal{A}_2$.

REFERENCES

1. B. J. Ball, *The sum of two solid horned spheres*, Ann. of Math. (2) **69** (1959), 253–257. MR **21** #321.

2. R. H. Bing, *Upper semicontinuous decompositions of* E^3, Ann. of Math. (2) **65** (1957), 363–374. MR **19,** 1187.

3. R. H. Fox and E. Artin, *Some wild cells and spheres in 3-dimensional space*, Ann. of Math. (2) **49** (1948), 979–990. MR **10,** 317.

4. N. Hosay, *The sum of a cube and a crumpled cube is* S^3, Abstract #607-17, Notices Amer. Math. Soc. **10** (1963), 666.

5. L. L. Lininger, *Some results on crumpled cubes*, Trans. Amer. Math. Soc. **118** (1965), 534–549. MR **31** #2717.

6. Joseph Martin, *The sum of two crumpled cubes*, Michigan Math. J. **13** (1966), 147–151. MR **32** #8324.

7. C. D. Sikkema, *A duality between certain spheres and arcs in* S^3, Trans. Amer. Math. Soc. **122** (1966), 399–415. MR **33** #7991.

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