REPRESENTATION OF LINEAR SETS AS CRITICAL SETS

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ABSTRACT. A class of linear sets investigated by Besicovitch and Taylor is related to the critical set of differentiable mappings of a specified degree of smoothness. An example is constructed to show that certain results on Hausdorff measure are nearly best-possible.

Let F be a compact set of real numbers and $[a, b]$ the smallest interval containing F. The complement $[a, b]\sim F$ is composed of a countable sequence of disjoint open intervals, of lengths l_n. We investigate sets F of Lebesgue measure 0 with the property that $\sum l_n^c < \infty$ for some c in (0, 1). These sets were considered by Besicovitch and Taylor in [1] but our theorems are in a different direction.

We require a class of functions C^β defined for each number $\beta > 1$: a real function f on an interval is of class C^β provided it is n times continuously differentiable, where $n < \beta \leqq n+1$, and $D^n f$ is of class $\mathrm{Lip}^{\beta-n}$. When $\beta = n+1$ this conflicts with the usual definition of C^{n+1}, but no confusion is to be expected; in fact by allowing a larger class C^{n+1} we obtain a slightly sharper result.

THEOREM 1. *Let f belong to C^β, let Z be the zero-set of Df, and let $F = f(Z)$. Then F has Hausdorff $1/\beta$-measure 0, and the lengths l_n fulfill the condition $\sum l_n^{1/\beta} < \infty$.*

THEOREM 2. *Conversely, let F be a compact set of Lebesgue measure 0, whose contiguous intervals fulfill the convergence condition above. Then $F = f(Z)$ for some function f in C^β for which $Df \geqq 0$ and whose zero-set Z has Lebesgue measure 0. When $\beta = n+1$, f can be made $n+1$ times continuously differentiable.*

NOTATION. The diameter of a set E is written $|E|$, and its Lebesgue measure $m(E)$. The modulus of continuity of a function f on a set T is defined for $u > 0$ as

$$w(u) = \sup |f(t_1) - f(t_2)| : |t_1 - t_2| \leqq u.$$

Then $w(u) \ll u^c$ defines the class Lip^c, $0 < c \leqq 1$.

1. The proof of Theorem 1 is largely a variant of Taylor's theorem,

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the object being to exploit the extra information on the highest-order derivative.

LEMMA 1. *Let f be k times continuously differentiable on an interval $[c, d]$ and let $Df, \cdots, D^k f$ vanish at least once in the interval. Then*

$$\int_c^d |Df(t)|\, dt \leqq (d-c)^{k-1} \int_0^{d-c} w(u)du,$$

where w is the modulus of continuity of $D^k f$.

PROOF. For $k=1$ and $Df(\xi)=0$, $c \leqq \xi \leqq d$, we have

$$\int_c^d |Df(t)|\, dt \leqq \int_0^{\xi-c} + \int_0^{d-\xi} w(u)du \leqq \int_0^{d-c} w(u)du.$$

Assuming the truth of the lemma for $k-1 \geqq 1$,

$$\int_c^d |D^2 f(t)|\, dt \leqq (d-c)^{k-2} \int_0^{d-c} w(u)du.$$

Because Df has a zero in $[c, d]$, $|Df| \leqq \int_c^d |D^2 f(t)| dt$ and the lemma follows from this.

To prove Theorem 1, we observe first that each interval $I=(t_1, t_2)$ contiguous to $f(Z)$ has the form $f(J)$ for some interval J contiguous to Z. Indeed, let $s_1 \in f^{-1}(t_1)$ and $s_2 \in f^{-1}(t_2)$ be so chosen that $|s_1-s_2|$ attains its minimum value. Then the interval J between s_1 and s_2 is mapped into I, and therefore onto I. Thus it is sufficient to prove that $\sum |f(J)|^{1/\beta} < \infty$, where the summation is extended to intervals J contiguous to Z.

First, let Z' be the derived set of Z and let J have at least one end point in Z'. Then $Df, \cdots, D^n f$ vanish there and by Lemma 1

$$|f(J)| = \left| \int_J Df \right| \leqq |J|^{n-1} \int_0^{|J|} w(u)du \ll |J|^{\beta}.$$

To treat the isolated points in Z, let J be an interval contiguous to Z', so that if J meets Z then $J \cap Z$ is discrete. Thus $J \cap Z$ can be enumerated $\cdots < z_{-1} < z_0 < z_1 < \cdots$, and we must estimate the sum $\cdots + |f(z_1) - f(z_0)|^{1/\beta} + \cdots$. The sequence $z_{-1} < z_0 < z_1 < \cdots$ can be arranged into disjoint blocks of exactly $n+1$ terms, with a possible remainder of at most n terms. By Rolle's Theorem we know that $D^2 f, \cdots, D^n f$ each have zeros on any interval $[z_i, z_{i+n}]$, whence $\int_{z_i}^{z_{i+n}} |Df| \ll |z_{i+n} - z_i|^{\beta}$. The same estimate can be made for the remainder allowed before, because one of the extreme terms is succeeded immediately by an element of Z'. Applying the inequality

$$\sum_0^n x_i^{1/\beta} \leqq (n+1)^{1-1/\beta}\left(\sum_0^n x_i\right)^{1/\beta},$$

we find that

$$\cdots + |f(z_1) - f(z_0)|^{1/\beta} + \cdots \ll |J|.$$

The estimation given applies to all but $2n$ intervals situated entirely to one side of Z', and the proof is complete. (The possibility that $Z' = \varnothing$ makes the last remark necessary.)

That F has Hausdorff $1/\beta$-measure 0 is proved very simply in [**1**], but we present a less elementary proof for a stronger conclusion.

LEMMA 2. *Let f be absolutely continuous on an interval $[c, d]$ and let $\int_A |Df| = 0$ for a closed subset $A \subseteq [c, d]$. Suppose that for every interval J contiguous to A,*

$$\int_J |Df| \ll |J|^\beta \quad \textit{for a certain real number } \beta > 1.$$

Then $f(A)$ is contained in $o(N)$ intervals of length $N^{-\beta}$, $N \to +\infty$.

PROOF. We shall replace f by a function g that coincides with f on A, and is again absolutely continuous. To do so we define Dg on the intervals J so that $\int_J Dg = \int_J Df$. Thus, when $J = (t_1, t_2)$, set

$$Dg(s) = c(s - t_1)^{\beta-1}, \quad t_1 < s \leqq \tfrac{1}{2}(t_1 + t_2),$$
$$Dg(s) = c(t_2 - s)^{\beta-1}, \quad \tfrac{1}{2}(t_1 + t_2) < s < t_2$$

for a constant c. Then in fact

$$c = 2^{\beta-1}(t_2 - t_1)^{-\beta}(f(t_2) - f(t_1)) \ll 1,$$

whence

$$Dg(s) \ll (\mathrm{dist}(s, A))^{\beta-1}.$$

Let I_N denote any of the intervals $[kN^{-1}, (k+1)N^{-1}]$ that meet A, so that $f(A) = g(A) \subseteq \bigcup g(I_N)$. On each I_N we have $|Dg| \ll N^{1-\beta}$, so that $|f(I_N)| \ll N^{-\beta}$. Of course, the number of intervals I_N is $O(N)$.

Fixing a number $\epsilon < 1$ we divide the intervals I_N into two classes.

(i) $m(I_N \cap A) > (1-\epsilon)N^{-1}$. In this event every point of I_N is within ϵN^{-1} of A, and this allows us to introduce a factor $\epsilon^{\beta-1}$ into the previous estimate of $|g(I_N)|$, still preserving the number of intervals I_N.

(ii) $m(I_N \cap A) \leqq (1-\epsilon)N^{-1}$. Let us write ν_N for the number of the intervals, and ν'_N for the number treated in (i). Then $m(A) \leqq N^{-1}\nu'_N + (1-\epsilon)N^{-1}\nu_N$. But because A is closed, $\nu_N + \nu'_N = Nm(A) + o(N)$, hence $\nu_N = o(N)$. Lemma 2 is an easy consequence of this fact and the estimate given in (i).

To obtain the result on the $1/\beta$-measure of $f(Z)$ we select $A = Z'$ and note that $Z \sim Z'$ is countable. It is worth remarking that if $Df \geqq 0$ and $m(Z) = 0$ then f is strictly monotone and $f(Z \sim Z')$ is the set of isolated points of $f(Z)$.

2. In this section we suppose that F is a set described in Theorem 2. Let g be absolutely continuous on $[a, b] \supseteq F$, and linear on each contiguous interval J, with derivative $|J|^{1/\beta - 1}$. The mapping inverse to g, say h, is increasing and continuous because $Dg > 0$ almost everywhere. But h is also absolutely continuous because it maps each (Lebesgue) null set onto a null set. Thus F is subject to the previous lemma, since $|g(J)| = |J|^{1/\beta}$ and $m(g(F)) = 0$.

In the proof of Theorem 2 we keep the function h, but regard it solely as a mapping of $g(F)$ onto F. We now extend h to a mapping of class C^β. Let χ be a function in $C^\infty[0, 1]$,

$$\chi(0) = 0, \qquad \chi(1) = 1, \qquad D\chi > 0 \text{ on } (0, 1), \qquad D^k\chi(0) = D^k\chi(1) = 0, \quad 1 \leqq k < \infty.$$

On each interval (t_1, t_2) contiguous to $g(F)$ we define

$$f(s) = h(t_1) + (t_2 - t_1)^\beta \chi(|s - t_1|/(t_2 - t_1)), \qquad t_1 < s < t_2.$$

Then $f(t_1+) = h(t_1)$, $f(t_2-) = h(t_1) + (t_2 - t_1)^\beta = h(t_2)$. When $1 \leqq k < \infty$,

$$D^k f(s) = (t_2 - t_1)^{\beta - k} D^k \chi(|s - t_1|/(t_2 - t_1)).$$

In particular the kth derivative of f, on the complement of $g(F)$, is uniformly bounded for $1 \leqq k \leqq \beta$.

Now f is absolutely continuous, for it is monotone-increasing and continuous, and preserves null sets. Hence its derivative is given by Df (extended to all of $h([a, b])$. Also, the functions $Df, \cdots, D^{n-1}f$ are continuous on $h[a, b]$, vanish on $h(F)$, and have uniformly bounded derivatives on the complement of $h(F)$. It follows that each is the derivative of its predecessor; for the same reasons $D^n f$ is the derivative of $D^{n-1}f$, and f is n times continuously differentiable. From the formula for $D^n f$, it vanishes continuously on $h(F)$, and when $\beta < n+1$, $D^n f$ satisfies a Lipschitz condition of order $\beta - n$, on the contiguous intervals. From these facts the Lipschitz condition for all of $h([a, b])$ is easily deduced.

To improve this result for $\beta = n+1$, we proceed as follows. Writing $l_1 \geqq l_2 \geqq \cdots \geqq l_n \geqq \cdots$ for the lengths of intervals I_n complementary to F, we find numbers $1 < c_1 < c_2 < \cdots < c_n \to +\infty$ such that

$\sum(c_n l_n)^{1/\beta}<\infty$. We then modify the function g, so that I_n is mapped onto an interval of length $(c_n l_n)^{1/\beta}$. The function h inverse to g is also modified and so ultimately is the function f (constructed with the aid of the auxiliary mapping χ). We consider in detail this function, $\tilde{f}$.

Writing (t_1, t_2) for the transform by g of the interval I_n, we have

$$t_2 = t_1 + (c_n l_n)^{1/\beta},$$
$$\tilde{f}(s) = h(t_1) + c_n^{-1}(t_2 - t_1)^{\beta}\chi(|s - t_2|/(t_2 - t_1)), \quad t_1 < s < t_2,$$
$$D^{n+1}\tilde{f}(s) = c_n^{-1} D^{n+1}\chi(|s - t_2|/(t_2 - t_1)).$$

Since the factor c^{-1} converges to 0 with the length of the interval (t_1, t_2), $\tilde{f}$ belongs to the conventional class C^{n+1}.

3. In this section we show that the vanishing of the $1/\beta$-measure of $f(Z)$ cannot be strengthened very much. Let q be a function on $(0, \infty)$ such that $q(t)$ and $t^{1/\beta}/q(t)$ are increasing, $\sum_1^\infty q(2^{-m})<\infty$.

Theorem 3. *There exists a function f satisfying the conditions of Theorem 1, for which $f=F(Z)$ has positive Hausdorff measure with respect to the function $\phi(t)=t^{1/\beta}/q(t)$.*

Choosing $q(t)=\log^2(t^{-1})$ for small t, we find that $F(Z)$ can have dimension $1/\beta$.

Proof. Without loss of generality we can suppose $\sum_1^\infty q(2^{-m})<1$. In each dyadic interval $[k2^{-m}, (k+1)2^{-m}]\subseteq[0, 1]$ we construct an interval centered at $(k+\frac{1}{2})2^{-m}$, of length $2^{-m}q(2^{-m})$. We remove all intervals defined for $m=1$, then all intervals defined for $m=2$ save those intersecting an interval already removed, and so on. The disjoint intervals selected form an open set W of measure $m(W)<1$. Let $Df=0$ on $Z=[0, 1]\sim W$, and on an interval I of W, let $Df=|I|^{\beta-1}$. Then $f(Z)$ is a set F, since the contiguous intervals have lengths $|I|^\beta$ corresponding to the components I of W.

Observe next that if $s_1<s_2$ and $s_1,s_2\in Z$, the $f(s_2)-f(s_1)\gg(s_2-s_1)^\beta q^\beta(s_2-s_1)$. Indeed (s_1, s_2) contains a dyadic interval $[k2^{-m}, (k+1)2^{-m}]$, with $2^{-m}\geqq\frac{1}{4}(s_2-s_1)$. The interval constructed in $[k2^{-m}, (k+1)2^{-m}]$ either belongs to W, or intersects a larger interval contained in W, of length $\geqq 2^{-m}q(2^{-m})\gg(s_2-s_1)q(s_2-s_1)$. In any case an interval of that length belongs entirely to $W\cap(s_1, s_2)$, whence the lower bound on $f(s_2)-f(s_1)$.

Let μ be the measure of Borel sets E defined by

$$\mu(E) = m(Z\cap f^{-1}(E)), \quad \mu(f(Z)) = m(Z) > 0.$$

The proof will be completed by showing that $\mu(I)\ll\phi(|I|)$ for all intervals I. Now I contains a subinterval I_0 with end points in $f(Z)$, such that $\mu(I_0)=\mu(I)$, and of course $\phi(|I_0|)\leqq\phi(|I|)$.

Let $I_0=f(J)$, for an interval J contiguous to W. Then $\mu(I_0)\leqq|J|$, while $|I_0|\gg|J|^\beta q^\beta(|J|)$. Thus

$$\mu(I_0) \ll |I_0|^{1/\beta}/q(|J|) \ll |I_0|^{1/\beta}/q(|I_0|),$$

because $|I_0|\ll|J|$.

Related questions in Euclidean space have been treated by Sard in [2].

References

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