AN INEQUALITY FOR THE RIEMANN-STIELTJES INTEGRAL[1]

RICHARD DARST AND HARRY POLLARD

ABSTRACT. Let g and h be real valued and continuous on the interval $[a, b]$, and suppose that the variation, $V[h]$, of h on $[a, b]$ is finite. By completely elementary methods, it is shown that $V[h] \cdot \sup_{a \leqq \alpha < \beta \leqq b}(g(\beta) - g(\alpha))$ is an upper bound for $\int_a^b (h - \inf h)dg$.

Several writers have recently obtained upper bounds for integrals of the form $\int_a^b hdg$, where h is of bounded variation on the interval $[a, b]$ and g is continuous there ([1], [2], [3, p. 573], [4]). It is our purpose to establish the following extension by completely elementary methods.

THEOREM. *If h is real and of bounded variation on the interval $[a, b]$ and g is real and continuous there, then*

$$\int_a^b hdg \leqq (\inf h)(g(b) - g(a)) + S[a, b]V[h],$$

where $V[h]$ is the total variation of h, and

$$S[a, b] = \sup_{a \leqq \alpha < \beta \leqq b} \int_\alpha^\beta dg.$$

We observe first that it is enough to prove the inequality in the case $\inf h = 0$, when it becomes

$$(*) \qquad \int_a^b hdg \leqq S[a, b]V[h].$$

For the general case can be obtained from (*) by replacing h in it by $h - \inf h$. Clearly we may also suppose that for some ξ, $h(\xi) = 0$. Since

$$\int_a^b hdg = \int_a^\xi hdg + \int_{-b}^{-\xi} h(-x)d[-g(-x)]$$

and $\sup_{-b \leqq \alpha < \beta \leqq -\xi}([-g(-\beta)] - [-g(-\alpha)]) = S[\xi, b]$, we need only show

Received by the editors October 29, 1969 and, in revised form, November 14, 1969.

AMS Subject Classifications. Primary 2646.

Key Words and Phrases. Riemann-Stieltjes integral, bounded variation, upper bound.

[1] This research was supported in part by the National Science Foundation under grant no. GP 9470.

(L) If $h \geqq 0$ and $h(b)=0$, then

$$\int_a^b hdg \leqq S[a,b]V[h].$$

PROOF OF (L). Assume $g(a)=0$. Let $\phi(t)=\inf_{\xi\in[a,t]}g(\xi)$ and let $\psi(t)=g(t)-\phi(t)=\sup_{\xi\in[a,t]}(g(t)-g(\xi))\leqq S[a, t]$.

Then ϕ is nonincreasing, $\phi(a)=0$, and $0\leqq\psi(t)\leqq S[a, t]$. Moreover

$$\int_a^b hdg = \int_a^b hd\phi + \int_a^b hd\psi \leqq 0 + \int_a^b hd\psi$$

$$= -\int_a^b \psi dh \leqq \|\psi\|_\infty V[h] \leqq S[a,b]V[h].$$

REFERENCES

1. T. Ganelius, *Un théorème taubérien pour la transformation de Laplace*, C. R. Acad. Sci. Paris **242** (1965), 719–721.

2. ———, *An inequality for Stieltjes integrals*, Proc. Fourteenth Scandinavian Math. Congress, Copenhagen, 1964.

3. P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964. MR **30** #1270.

4. R. Wüst, *Beweis eines Lemmas von Ganelius*, Jber. Deutsch. Math.-Verein. **71** (1969), 229–230.

PURDUE UNIVERSITY, LAFAYETTE, INDIANA 47907