AN INEQUALITY FOR THE
RIEMANN-STIELTJES INTEGRAL

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Abstract. Let $g$ and $h$ be real valued and continuous on the interval $[a, b]$, and suppose that the variation, $V[h]$, of $h$ on $[a, b]$ is finite. By completely elementary methods, it is shown that $V[h] \cdot \sup_{a \leq \alpha < \beta \leq b} (g(\beta) - g(\alpha))$ is an upper bound for $\int_a^b (h - \inf h)dg$.

Several writers have recently obtained upper bounds for integrals of the form $\int_a^b hdg$, where $h$ is of bounded variation on the interval $[a, b]$ and $g$ is continuous there ([1], [2], [3, p. 573], [4]). It is our purpose to establish the following extension by completely elementary methods.

Theorem. If $h$ is real and of bounded variation on the interval $[a, b]$ and $g$ is real and continuous there, then

$$\int_a^b hdg \leq (\inf h)(g(b) - g(a)) + S[a, b]V[h],$$

where $V[h]$ is the total variation of $h$, and

$$S[a, b] = \sup_{a \leq \alpha < \beta \leq b} \int_{\alpha}^{\beta} dg.$$

We observe first that it is enough to prove the inequality in the case $\inf h = 0$, when it becomes

$$(*) \quad \int_a^b hdg \leq S[a, b]V[h].$$

For the general case can be obtained from $(*)$ by replacing $h$ in it by $h - \inf h$. Clearly we may also suppose that for some $\xi$, $h(\xi) = 0$. Since

$$\int_a^b hdg = \int_a^{\xi} hdg + \int_{-x}^{-\xi} h(-x)d[-g(-x)]$$

and $\sup_{a \leq \alpha < \beta \leq \xi} ([-g(-\beta)] - [-g(-\alpha)]) = S[\xi, b]$, we need only show...
(L) If $h \geq 0$ and $h(b) = 0$, then

$$\int_a^b h dg \leq S[a, b]V[h].$$

**Proof of (L).** Assume $g(a) = 0$. Let $\phi(t) = \inf_{t \in [a, t]} g(\xi)$ and let $\psi(t) = g(t) - \phi(t) = \sup_{t \in [a, t]} (g(t) - g(\xi)) \leq S[a, t]$.

Then $\phi$ is nonincreasing, $\phi(a) = 0$, and $0 \leq \psi(t) \leq S[a, t]$. Moreover

$$\int_a^b h dg = \int_a^b h d\phi + \int_a^b h d\psi \leq 0 + \int_a^b h d\psi$$

$$= -\int_a^b \psi dh \leq \|\psi\|_\infty V[h] \leq S[a, b]V[h].$$

**References**


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