ON MULTIPLIERS AND ORDER-BOUNDED OPERATORS IN C^*-ALGEBRAS

TAGE BAI ANDERSEN

ABSTRACT. It is shown that the selfadjoint multipliers from a sub-C^*-algebra to the bigger C^*-algebra are exactly the order-bounded operators. As a corollary we get a characterization of the relative commutant of a sub-C^*-algebra with unit.

If A is a C^*-algebra, we shall denote the selfadjoint part of A by A_{sa}, the positive part of A by A^+, and the unit ball of A by A_1. If A and B are C^*-algebras a linear map T from A into B will be called selfadjoint if $T(a^*)=(Ta)^*$ for all a in A.

DEFINITION 1. Let A be a sub-C^*-algebra of the C^*-algebra B. A selfadjoint linear map T from A into B will be called *order-bounded* if there is a nonnegative real number λ such that

$$-\lambda a \leqq Ta \leqq \lambda a, \qquad \text{all } a \in A^+.$$

REMARK 1. If T is order-bounded then T is automatically continuous. In fact, if $a\in A^+$ then $\|Ta\|\leqq\lambda\|a\|$. If $a\in A$ then $a=a_1-a_2+i(a_3-a_4)$, where $a_i\in A^+$ and $\|a_i\|\leqq\|a\|$ for $i=1,\cdots,4$. Then $\|Ta\|\leqq 4\lambda\cdot\|a\|$.

DEFINITION 2. Let A be a sub-C^*-algebra of the C^*-algebra B. A bounded linear map T from A into B is said to *act as a multiplier* if

$$T(ab) = (Ta)b = a(Tb), \qquad \text{all } a, b \in A.$$

(A multiplier is a double centralizer in the sense of [3].)

THEOREM 1. *Let A be a sub-C^*-algebra of the C^*-algebra B. Let T be a bounded selfadjoint linear map from A into B. Then the following conditions are equivalent*:

(i) *T acts as a multiplier.*

(ii) *T is order-bounded.*

PROOF. (i)$\Rightarrow$(ii). Let $x\in A_1^+$, and let $0<\alpha<1$. Since T acts as a multiplier: $x^\alpha(Tx^{1-\alpha})=Tx=(Tx^{1-\alpha})x^\alpha$, and hence x^α and $Tx^{1-\alpha}$ commute. Let f be a positive linear functional on B. Then $(Tx^{1-\alpha})x^\alpha\leqq\|Tx^{1-\alpha}\|x^\alpha$ by Gelfand theory and

Received by the editors December 23, 1969.

AMS Subject Classifications. Primary 4665.

Key Words and Phrases. Multiplier, order-bounded operator, relative commutant.

$$\begin{aligned} f(Tx) &= f((Tx^{1-\alpha})x^{\alpha}) \leqq f(\|Tx^{1-\alpha}\|x^{\alpha}) \\ &\leqq \|T\|f(x^{\alpha}) \to \|T\|f(x) \quad \text{for } \alpha \to 1. \end{aligned}$$

(Since $x^{\alpha} \to x$ in norm by Gelfand theory.) Analogously, $-\|T\|f(x) \leqq f(Tx)$, and since the order of B is completely determined by the positive linear functionals [2, 2.6.2] we get $-\|T\|x \leqq Tx \leqq \|T\|x$ and T is order-bounded.

(ii)⇒(i). By replacing T by $(2\lambda)^{-1}(\lambda I + T)$ we may assume: $0 \leqq Ta \leqq a$, for all $a \in A^+$.

Let π be any faithful representation of B, e.g. the universal representation of B [2, 2.7.6]. By considering $\pi(A)$, $\pi(B)$ and $\pi \circ T \circ \pi^{-1}$ we may assume A and B to be concrete C^*-algebras acting on a Hilbert space H such that B is nondegenerate.

We notice that if $\{a_\gamma\}_{\gamma \in G}$ is a net from A_1^+ tending weakly to 0 then for all $\gamma \in G$ and $\xi \in H$

$$0 \leqq (Ta_\gamma \xi, \xi) \leqq (a_\gamma \xi, \xi)$$

and hence $(Ta_\gamma \xi, \xi) \to 0$. By a result of Kadison [4, Remark 2.2.3] T has an extension $\overline{T}$ to the weak closure $\overline{A}$ of A into $\overline{B}$ such that $\overline{T}$ is weakly continuous on the unit ball $\overline{A}_1$ of $\overline{A}$.

If $a \in \overline{A}_1^+$ there is by Kaplansky's density theorem [1, Theorem 3, p. 46] a net $\{a_\gamma\}_{\gamma \in G} \subseteq A_1^+$ such that $a_\gamma \to a$ strongly. For each $\gamma \in G$ and $\xi \in H$ we get

$$0 \leqq (Ta_\gamma \xi, \xi) \leqq (a_\gamma \xi, \xi)$$

and since $\overline{T}$ is weakly continuous on $\overline{A}_1$, $0 \leqq (\overline{T}a\xi, \xi) \leqq (a\xi, \xi)$, and $\overline{T}$ is order-bounded.

Since $\overline{B}$ (resp. $\overline{A}$) is weakly closed $\overline{B}$ (resp. $\overline{A}$) has a unit e_1 (resp. e_2) [1, Theorem 2, p. 44].

Since $\overline{B}$ is nondegenerate e_1 is the identity operator on H, while e_2 is a projection in $\overline{B}$, and $\overline{A}$ is a von Neumann algebra on the Hilbert space $e_2 H$.

Let p be a projection in $\overline{A}$. Then $0 \leqq \overline{T}p \leqq p$ and so for all $\xi \in H$

$$\begin{aligned} 0 &\leqq ((\overline{T}p)^{1/2}(e_1 - p)\xi, (\overline{T}p)^{1/2}(e_1 - p)\xi) \\ &\leqq ((\overline{T}p)(e_1 - p)\xi, (e_1 - p)\xi) \\ &\leqq (p(e_1 - p)\xi, (e_1 - p)\xi) = 0. \end{aligned}$$

This means that $(\overline{T}p)^{1/2}(e_1 - p) = 0$ and hence $(\overline{T}p)(e_1 - p) = 0$ i.e. $\overline{T}p = (\overline{T}p) \cdot p$.

Since $\overline{T}p$ is selfadjoint we also get $\overline{T}p = p(\overline{T}p)$. Now we get

$$(\overline{T}e_2)p = (\overline{T}p)\cdot p + (\overline{T}(e_2 - p))p$$
$$= \overline{T}p + (\overline{T}(e_2 - p))(e_2 - p)p$$
$$= \overline{T}p$$

and analogously $p(\overline{T}e_2) = \overline{T}p$.

Since $\overline{T}$ is norm-continuous on $\overline{A}_{sa}$ and $\overline{T}p = (\overline{T}e_2)p = p(\overline{T}e_2)$ for all projections p in $\overline{A}$ it follows from the spectral theorem that

$$\overline{T}a = (\overline{T}e_2)a = a(\overline{T}e_2), \qquad \text{for all } a \in \overline{A}_{sa},$$

and hence for all $a \in \overline{A}$. Especially, $\overline{T}e_2 \in A'$.

Now let $a, b \in A$. Then

$$T(ab) = \overline{T}(ab) = (\overline{T}e_2)(ab) = ((\overline{T}e_2)(a))b = (\overline{T}a)b = (Ta)b$$

and since $\overline{T}e_2 \in A'$ we get analogously $T(ab) = a(Tb)$, and the theorem is proved.

If A has a unit we can obtain the following characterization of the relative commutant of A in B:

COROLLARY 1. *Let A be a sub-C^*-algebra of the C^*-algebra B, and assume that A has a unit e. Let T be a bounded selfadjoint linear operator from A into B. Then the following conditions are equivalent*:

(i) *T is order-bounded.*

(ii) *T is multiplication by a selfadjoint element of $A' \cap B$ (e.g. Te).*

PROOF. (i)$\Rightarrow$(ii). By Theorem 1 T acts as a multiplier. Then for all a in A: $Ta = T(ea) = (Te)a$ and $Ta = T(ae) = a(Te)$. Hence $Te \in A' \cap B$, and the result follows.

(ii)$\Rightarrow$(i). This follows either directly from Theorem 1 or by the observation that if b is a selfadjoint element from $A' \cap B$ and $a \in A^+$ then: $-\|b\|a \leqq ba \leqq \|b\|a$.

If we specialize $A = B$ we get the following well-known result

COROLLARY 2. *Let A be a C^*-algebra with unit. Let T be a bounded selfadjoint linear operator in A. Then T is order-bounded if and only if T is multiplication by a selfadjoint central element in A.*

COROLLARY 3. *Let A and B be C^*-algebras, Φ a $*$-homomorphism of A into B, and T a bounded selfadjoint linear operator from A into B. If there is a nonnegative real number λ such that*

$$-\lambda\Phi(a) \leqq Ta \leqq \lambda\Phi(a), \qquad \textit{all } a \in A^+,$$

then

$$T(ab) = (Ta)\Phi(b) = \Phi(a)(Tb), \qquad \textit{all } a, b \in A.$$

*Moreover, if A has a unit e then $Te \in \Phi(A)' \cap B$ and $Ta = (Te)\,\Phi(a)$. Especially, T is a *-homomorphism if and only if Te is a projection in $B \cap \Phi(A)'$.*

PROOF. Since the kernel of Φ is positively generated T is 0 on $\ker(\Phi)$. Hence the map $\Psi : \Phi(A) \to B$ is well defined by $\Psi(\Phi(a)) = Ta$, all $a \in A$, as a linear operator. Since T is selfadjoint Ψ is selfadjoint. $\Phi(A)$ is a sub-C^*-algebra of B, and since $\Phi(A^+) = \Phi(A)^+$, Ψ is order-bounded. By Remark 1 Ψ is continuous. By Theorem 1, Ψ acts as a multiplier, and hence for all a, $b \in A$

$$\begin{aligned} T(ab) &= \Psi(\Phi(ab)) = \Psi(\Phi(a)\Phi(b)) \\ &= \Psi(\Phi(a))\Phi(b) = (Ta)\Phi(b) \end{aligned}$$

and analogously $T(ab) = \Phi(a)(Tb)$.

Now the second part follows easily from the first part.

ACKNOWLEDGEMENTS. We want to thank G. Kjærgård Pedersen and E. Størmer for helpful comments.

REFERENCES

1. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, Cahiers Scientifiques, fasc. 25, Gauthier-Villars, Paris, 1957. MR **20** #1234.

2. ———, *Les C*-algèbres et leurs représentations*, Cahiers Scientifiques, fasc. 29, Gauthier-Villars, Paris, 1964. MR **30** #1404.

3. B. E. Johnson, *An introduction to the theory of centralizers*, Proc. London Math. Soc. (3) **14** (1964), 299–320. MR **28** #2450.

4. R. V. Kadison, *Unitary invariants for representations of operator algebras* Ann. of Math. (2) **66** (1957), 304–379. MR **19**, 665.

UNIVERSITY OF OSLO, OSLO, NORWAY

UNIVERSITY OF AARHUS, AARHUS, DENMARK