CHARACTERIZATION OF THE FINITE PARTITION PROPERTY FOR A COLLECTION OF UNIVERSAL SUBCONTINUA

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Let X be a Hausdorff space and $A \subset X$ a continuum. A is said to be a Universal Subcontinuum (USC) if $A \cap B$ is connected for every continuum $B \subset X$. Let α be a collection of USC's of a Hausdorff space. Then α is said to have the finite partition property if α has a decomposition into a finite number of subcollections each having the finite intersection property. A result due to W. J. Gray [1] can be easily modified to show that in a Hausdorff space, a collection of USC's has the finite intersection property if every pair has a common point. Other properties of USC's are given in [2] and [3].

THEOREM. *Let α be a collection of USC's of a Hausdorff space. Then the following statements are equivalent.*

(1) *α has the finite partition property.*

(2) *There exist integers p, q with $p \geqq q \geqq 2$ such that for every p elements of α, at least q of them have a common point.*

(3) *α has no infinite pairwise disjoint subcollection.*

PROOF. (1) implies (2) and (2) implies (3) are obvious. Condition (2) is used in [2] to obtain a result which states that the maximal number of subcollections required for the partition is $p-q+2$. We now prove that (3) implies (1). The proof is by contradiction; we assume that α is a collection of USC's of a Hausdorff space with no infinite pairwise disjoint subcollection, but that α does not have the finite partition property.

Let $I(\alpha)=\{\beta \subset \alpha \mid \beta \text{ is pairwise disjoint}\}$. Let $\beta_1, \beta_2 \in I(\alpha)$. We say $\beta_1 \leqq \beta_2$ if $\beta_1 \subset \beta_2$. Then it is clear that $\leqq$ defines a partial order on $I(\alpha)$. Let $\{\beta_j \mid j \in J\}$ be a totally ordered subset of $I(\alpha)$. Then define $\beta = \bigcup_{j\in J} \beta_j$. We show $\beta \in I(\alpha)$. Let $H, G \in \beta$, then there exist $j_1, j_2 \in J$ such that $H \in \beta_{j_1}$ and $G \in \beta_{j_2}$. We may assume $\beta_{j_1} \subset \beta_{j_2}$ which implies that $H, G \in \beta_{j_2}$ and that $H \cap G = \emptyset$ if $H \neq G$. Thus β is pairwise disjoint and $\beta \in I(\alpha)$. It is clear that $\beta \geqq \beta_j$ for every $j \in J$ and hence β is an upper bound for the chain. Thus every pairwise disjoint subcollection of α is a subset of some maximal element of $I(\alpha)$.

Let $\alpha = \alpha_0$; we define the following subcollections inductively: $\beta_i, \alpha_i^1, \alpha_i^2, \beta_i^1, \alpha_i^3, \alpha_{i+1}$.

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(1) β_i is a maximal element of $I(\alpha_i)$ with card $\beta_i \geqq 2$.

(2) $\alpha_i^1 = \{H \in \alpha_i \mid$ there exist $B_1, B_2 \in \beta_i$ such that $B_1 \cap B_2 = \emptyset$ and $H \cap B_1 \neq \emptyset \neq H \cap B_2\}$.

(3) $\alpha_i^2 = \{H \in \alpha_i - \alpha_i^1 \mid$ there exists $B_1, B_2 \in \beta_i$ and $G \in \alpha_i - \alpha_i^1$ such that $B_1 \cap B_2 = \emptyset$, $H \cap B_1 \neq \emptyset \neq G \cap B_2$, and $H \cap G \neq \emptyset\}$.

(4) $\beta_i^1 = \{B \in \beta_i \mid$ there exist $G_B, H_B \in \alpha_i - \alpha_i^1 - \alpha_i^2$ such that $G_B \cap B \neq \emptyset \neq H_B \cap B$ and $G_B \cap H_B = \emptyset\}$.

(5) $\alpha_i^3 = \{H \in \alpha_i - \alpha_i^1 - \alpha_i^2 \mid$ there exists $B \in \beta_i - \beta_i^1$ such that $H \cap B \neq \emptyset\}$.

(6) $\alpha_{i+1} = \alpha_i - \alpha_i^1 - \alpha_i^2 - \alpha_i^3$.

We proceed as follows: Since α_0 does not have the finite intersection property, there exist $H, G \in \alpha_0$ such that $H \cap G = \emptyset$. Then there exists a maximal element $\beta_0 \in I(\alpha_0)$ with $\{H, G\} \leqq \beta_0$. We now show that α_0^1 and α_0^2 have the finite partition property.

We associate each element of α_0^1 with the disjoint pair $B_1, B_2 \in \beta_0$ given in the definition of α_0^1 and then show that the collection of all elements associated with the same pair has the finite intersection property. Let $H_1, H_2 \in \alpha_0$; $B_1, B_2 \in \beta_0$; $H_1 \cap B_1 \neq \emptyset \neq H_1 \cap B_2$; $H_2 \cap B_1 \neq \emptyset \neq H_2 \cap B_2$; and $B_1 \cap B_2 = \emptyset$. Suppose $H_1 \cap H_2 = \emptyset$. Then $B_1 \cup H_1 \cup B_2$ and $B_1 \cup H_2 \cup B_2$ are USC's but $(B_1 \cup H_1 \cup B_2) \cap (B_1 \cup H_2 \cup B_2) = B_1 \cup B_2 = B_1 \mid B_2$. The contradiction shows $H_1 \cap H_2 \neq \emptyset$. By hypothesis, card β_0 is finite and it is clear that α_0^1 has the finite partition property. The proof that α_0^2 has the finite partition property is similar.

Now suppose that $\beta_0^1 = \emptyset$. Then since β_0 is maximal, every $H \in \alpha_0$ intersects some $B \in \beta_0$. We then decompose $\alpha_0 - \alpha_0^1 - \alpha_0^2$ into at most card β_0 subcollections by associating each element with the unique $B \in \beta_0$ which it intersects. Since $\beta_0^1 = \emptyset$, every pair associated with the same B have a common point. Thus $\alpha_0 - \alpha_0^1 - \alpha_0^2$ has the finite partition property and hence so does α_0. The contradiction shows $\beta_0^1 \neq \emptyset$. It is clear that α_0^3 has the finite partition property and that $\{G_B, H_B \mid B \in \beta_0^1\} \in I(\alpha_1)$. We choose a maximal element $\beta_1 \in I(\alpha_1)$ such that $\{G_B, H_B \mid B \in \beta_0^1\} \leqq \beta_1$. It α_1 has the finite partition property, then so does $\alpha_0 = \alpha_1 \cup \alpha_0^1 \cup \alpha_0^2 \cup \alpha_0^3$. Therefore we assume that α_1 does not have the finite partition property, and it is clear that this argument can be repeated inductively.

We now show that the sequence $\{\beta_i\}$ satisfies the following property:

(p) Suppose i, j, and k are integers such that $i \geqq j$ and $i \geqq k$. Let $B_i \in \beta_i$, $B_j \in \beta_j$, and $B_k \in \beta_k$. Then $B_i \cap B_j \neq \emptyset \neq B_i \cap B_k$ implies $B_j \cap B_k \neq \emptyset$.

Assume $B_j \cap B_k = \emptyset$. If $i = j$, then $B_i = B_j$ and $B_j \cap B_k \neq \emptyset$. Thus

$i \neq j$ and similarly, we have $i \neq k$. If $j = k$, then $B_j \cap B_k = \emptyset$ implies $B_i \in \alpha_j^1$. Thus $j \neq k$ and we may assume $i > j > k$. Since $B_j \in \alpha_j \subset \alpha_k$ and β_k is a maximal element of $I(\alpha_k)$, there exists $H_k \in \beta_k$ such that $B_j \cap H_k \neq \emptyset$ and $H_k \cap B_k = \emptyset$. But this implies $B_j \in \alpha_k^2$. The contradiction shows $B_j \cap B_k \neq \emptyset$.

Let $\gamma_i = \bigcup_{B \in \beta_i} B$. We now show $\bigcap_{i=0}^{\infty} \gamma_i \neq \emptyset$. Since each γ_i is compact, it suffices to show $\bigcap_{i=0}^{k} \gamma_i \neq \emptyset$ in every $k \geqq 0$. Let $B_k \in \beta_k$. Then in every $i \leqq k$, we have $B_k \in \alpha_k \subset \alpha_i$ and since β_i is a maximal element of $I(\alpha_i)$, there exists $B_i \in \beta_i$ such that $B_k \cap B_i \neq \emptyset$. But property (p) obviously implies $\bigcap_{i=0}^{k} B_i \neq \emptyset$ and thus $\bigcap_{i=0}^{\infty} \gamma_i \neq \emptyset$.

Let $x \in \bigcap_{i=0}^{\infty} \gamma_i$. Then for every $i \geqq 0$, there exists $D_i \in \beta_i$ such that $x \in D_i$. We now define a collection of USC's $\{E_i \mid i \geqq 0\}$ as follows:

(i) Since card $\beta_0 \geqq 2$, there exists $E_0 \in \beta_0$ such that $E_0 \cap D_0 = \emptyset$.

(ii) For every $i \geqq 0$, $D_{i+1} \in \alpha_i - \alpha_i^1 - \alpha_i^2 - \alpha_i^3$ and $D_{i+1} \cap D_i \neq \emptyset$ imply $D_i \in \beta_i^1$. Therefore, there exists $E_{i+1} \in \beta_{i+1}$ such that $E_{i+1} \cap D_i \neq \emptyset$ but $E_{i+1} \cap D_{i+1} = \emptyset$.

We now show that the infinite collection $\{E_i \mid i \geqq 0\}$ is pairwise disjoint. The proof is by induction. We have $E_0 \cap D_0 = \emptyset$, $E_1 \cap D_0 \neq \emptyset$, $D_1 \cap D_0 \neq \emptyset$, and $E_1 \cap D_1 = \emptyset$. Therefore $\{E_0, E_1, D_1\}$ is pairwise disjoint. Now assume $\{E_0, E_1, \cdots, E_k, D_k\}$ is pairwise disjoint. Then since $D_{k+1} \cap D_k \neq \emptyset$, $E_{k+1} \cap D_k \neq \emptyset$, and $E_{k+1} \cap D_{k+1} = \emptyset$, property (p) implies $\{E_0, E_1, \cdots, E_k, E_{k+1}, D_{k+1}\}$ is pairwise disjoint. By induction, $\{E_i \mid i \geqq 0\}$ is an infinite pairwise disjoint subcollection of α. This contradiction completes the proof.

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