ON A FINITE DIMENSIONAL QUASI-SIMPLE MODULE

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1. If R is a ring and M is a (right) R-module such that $MR \neq 0$, then M is said to be *quasi-simple* provided that

(Q1) the endomorphism (module endomorphism) ring of the quasi-injective hull of M is a division ring,

(Q2) if N is a nonzero submodule of M then there is a nonzero endomorphism f of M such that $f(M) \subseteq N$.

Any simple module is clearly quasi-simple, however a quasi-simple module is not necessarily simple. For example, if R is a semiprime ring with a uniform right ideal U such that the right singular ideal of R is zero then the regular R-module U is quasi-simple since the endomorphism ring of its quasi-injective hull is a division ring [**5**, 1.7, p. 263] and for any nonzero submodule N of U there is $x \in N$ such that $0 \neq xU \subseteq N$. Let us say that a quasi-simple module is *finite dimensional* if its quasi-injective hull is a finite dimensional vector space over its endomorphism ring. The main results in this paper are the following: Let M be a faithful quasi-simple R-module for some ring R and let $\tilde{M}$ be the quasi-injective hull of M. Let $D = \operatorname{Hom}_R(\tilde{M}, \tilde{M})$ and if N is a nonzero submodule of M then define $K(N) = \operatorname{Hom}_R(N, N)$. Then for any nonzero submodule N of M, $K(N)$ is a right order in D, $\tilde{M} = DN$ and if $\{m_i\}_{i=1}^n$ is a finite sequence of D-linearly independent elements in $\tilde{M}$ and if $\{y\}_{i=1}^n$ is a finite sequence in M, then there exist $r \in R$, $0 \neq k \in \operatorname{Hom}_R(M, M)$ such that $0 \neq k|_N \in K(N)$ and $m_i r = ky_i$ for all $1 \leqq i \leqq n$. If, in addition, the singular submodule of M is zero then, for any large right ideal B of R one can choose $r \in B$ and $0 \neq k \in \operatorname{Hom}_R(M, M)$ such that $0 \neq k|_N \in K(N)$ and $m_i r = ky_i$ for all $1 \leqq i \leqq n$. In case the set $\{m_1, m_2, \cdots, m_n\}$ is a basis for $\tilde{M}$ then r is a regular element. If R is a right Goldie prime ring and U is a uniform right ideal then U is a finite dimensional quasi-simple module. Conversely, if R has a faithful finite dimensional quasi-simple right module M then R is a right Goldie prime ring, and M is isomorphic to a uniform right ideal of R. Hence our theory above provides a new proof for Theorem 10 of [**3**, p. 603]. Let D_n be the $n \times n$ matrix ring representing $\operatorname{Hom}_D(\tilde{M}, \tilde{M})$ relative to the basis $\{m_1, m_2, \cdots, m_n\}$ and $K(N)_n$ be the $n \times n$ matrix ring over $K(N)$. Let R^* be the subring of D_n which represents R as a ring of linear transformations on $\tilde{M}$ over D relative to the same basis.

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Then $K(N)_n \cap R^*$ is a right order in D_n. We identify D_n and $\mathrm{Hom}_D(\tilde{M}, \tilde{M})$. For each $d \in D$, let $\bar{d}$ be a matrix in D_n such that $m_i \bar{d} = dm_i$ for all $1 \leqq i \leqq n$. Let $[K(N)] = \{\bar{d} \in D_n \mid d \in K(N)\}$. Then $[K(N)] \cap R^*$ is a right order in $\overline{D}$, where $\overline{D} = \{\bar{d} \in D_n \mid d \in D\}$. As one can see easily our theory here is a generalization of the density theorem for simple modules (refer to [**4**] and [**7**]).

2.0. Let R be a ring and M be an R-module (by an R-module we always mean a right R-module). We denote by $\hat{M}$ and $\tilde{M}$ respectively the injective hull and the quasi-injective hull of M. If S and T are nonempty subsets of M and R respectively, we define $S^r = \{r \in R \mid Sr = 0\}$, $T^r = \{r \in R \mid Tr = 0\}$, and $T^l = \{m \in M \mid mT = 0\}$. If $S = \{s\}$ and $T = \{t\}$ are singleton sets, we let $(s)^r = S^r$, $(t)^r = T^r$, and $(t)^l = T^l$. If M is a quasi-simple R-module and A and B are ideals of M such that $MAB = 0$ then either $MA = 0$ or $MB = 0$. Otherwise, one can find a nonzero $f \in \mathrm{Hom}_R(M, M)$ such that $f(M) \subseteq MA$ and hence $f(MB) = f(M)B \subseteq MAB = 0$. Since the kernel of f is zero this implies that $MB = 0$ which is absurd. Therefore M^r is a prime ideal. We also note here that any pair of nonzero submodules of $\tilde{M}$ has a nonzero intersection since $\mathrm{Hom}_R(\tilde{M}, \tilde{M})$ is a division ring.

2.1. Proposition. *Let M be an R-module for some ring R and let $D = \mathrm{Hom}_R(\tilde{M}, \tilde{M})$. Then $\tilde{M} = DM$.*

Proof. Clearly $DM \subseteq \tilde{M}$. Let $\Lambda = \mathrm{Hom}_R(\hat{M}, \hat{M})$. Then by [**2**, 1, p. 23], $\tilde{M} = \Lambda M$. If $x \in \tilde{M}$, then $x = \sum_{i=1}^n f_i(m_i)$ for some positive integer n where $f_i \in \Lambda$ and $m_i \in M$ for all $1 \leqq i \leqq n$. Let $d_i = f_i|_{\tilde{M}}$. Since $\Lambda\tilde{M} = \tilde{M}$, $d_i \in D$, so $x \in DM$. Therefore $\tilde{M} \subseteq DM$.

2.2. Proposition. *Let M be a quasi-simple R-module for a ring R. Then, for any nonzero submodule N of M, $\tilde{N} = \tilde{M} = DN$ where $D = \mathrm{Hom}_R(\tilde{M}, \tilde{M})$.*

Proof. Since D is a division ring, M is an essential extension of N. Therefore $\hat{N} = \hat{M}$. Let $\Lambda = \mathrm{Hom}_R(\hat{M}, \hat{M})$. Then $\Lambda N = \tilde{N}$ by [**2**, 1, p. 23], and hence $DN \subseteq \tilde{N}$. By (Q2), there is a nonzero $d \in D$ such that $d(M) \subseteq N$. Hence $Dd(M) \subseteq DN \subseteq \tilde{N}$. However $Dd = D$, since D is a division ring. Thus $\tilde{M} = DM = Dd(M) \subseteq DN \subseteq \tilde{N}$. Since $\tilde{N} \subseteq \tilde{M}$ always, this proves that $\tilde{M} = \tilde{N} = DN$.

2.3. Proposition. *Let M be a quasi-simple R-module for some ring R. If N is any nonzero submodule of M, then $\tilde{M}^r = N^r$.*

Proof. By 2.2. $\tilde{M} = \tilde{N} = DN$, where $D = \mathrm{Hom}_R(\tilde{M}, \tilde{M})$ is a division ring. It follows immediately that $N^r = (DN)^r = \tilde{M}^r$.

2.4. THEOREM. *Let M be a quasi-simple R-module for some ring R and let $D=\mathrm{Hom}_R(\tilde{M}, \tilde{M})$. If N is a nonzero submodule of M and if $K(N)=\mathrm{Hom}_R(N, N)$, then $K(N)$ is a right order in D.*

PROOF. Let $0\neq d\in D$ and let $N(d)=\{m\in M \mid d(m)\in N\}$. Then $N(d)$ is a nonzero submodule of M. Let $N_1=N\cap N(d)$. Since $N_1\neq 0$, there is $f\neq 0$ in $\mathrm{Hom}_R(M, M)$ such that $f(M)\subseteq N_1$ by (Q2). Hence $df(M)\subseteq d(N_1)\subseteq N$ and $df|_N\in K(N)$. This proves that $K(N)$ is a right order in D.

2.5. PROPOSITION. *Let M be a quasi-simple R-module for some ring R and let $D=\mathrm{Hom}_R(\tilde{M}, \tilde{M})$. If $\{x_i\}_{i=1}^n$ is a finite subset of $\tilde{M}$ and $y\in\tilde{M}$, then y is a D-linear combination of $\{x_i\}_{i=1}^n$ if and only if $\bigcap_{i=1}^n (x_i)^r\subseteq(y)^r$.*

PROOF. If y is a D-linear combination of $x_1, \cdots, x_n$, then clearly $\bigcap_{i=1}^n (x_i)^r\subseteq(y)^r$. Now, by [5, 2.2, p. 263], $(Dx_1+Dx_2+\cdots+Dx_n)^{rl}=Dx_1+Dx_2+\cdots+Dx_n$. If $\bigcap_{i=1}^n (x_i)^r\subseteq(y)^r$ then $(Dx_1+Dx_2+\cdots+Dx_n)^r=\bigcap_{i=1}^n(x_i)^r\subseteq(y)^r$ and $y\in(y)^{rl}\subseteq(Dx_1+Dx_2+\cdots+Dx_n)^{rb}=Dx_1+Dx_2+\cdots+Dx_n$.

2.6. THEOREM. *Let R be a ring and M be a quasi-simple R-module. Let $D=\mathrm{Hom}_R(\tilde{M}, \tilde{M})$ and let $K(N)=\mathrm{Hom}_R(N, N)$ where N is a nonzero submodule of M. If $\{x_i\}_{i=1}^n$ is a finite D-linearly independent subset of $\tilde{M}$ and $\{y_i\}_{i=1}^n$ is a sequence in M then there is $r\in R$ and $0\neq k\in\mathrm{Hom}_R(M, M)$ such that $k|_N\in K(N)$ and $x_ir=ky_i$ for all $1\leqq i\leqq n$.*

PROOF. Define $I_j=\bigcap_{i=1;i\neq 1}^n (x_i)^r$. The $x_jI_j\neq 0$ for any $1\leqq j\leqq n$. For, if $x_jI_j=0$ for some j, then $x_j\in\sum_{i=1;i\neq j}^n Dx_i$ by 2.5. and the set $\{x_i\}_{i=1}^n$ would not be linearly independent. Hence $x_jI_j\cap N$ is a nonzero submodule of M. By (Q2), there is a nonzero $f_j\in\mathrm{Hom}_R(M, M)$ such that $f_j(M)\subseteq x_jI_j\cap N$. Hence $f_j(y_j)\in x_jI_j\cap N$, and $0\neq f_j|_N\in K_R(N)$. Thus, if $y_j\neq 0$, there exists $a_j\in I_j$ such that $x_ja_j=f_jy_j\neq 0$. Without loss of generality, we may assume that $y_j\neq 0$ for $1\leqq j\leqq m$ and $y_j=0$ for $m<j\leqq n$. Then $(\bigcap_{j=1}^m f_jy_jR)\cap N$ is a nonzero submodule of N. Hence there is $0\neq k_0\in\mathrm{Hom}_R(M, M)$ such that $k_0(M)\subseteq(\bigcap_{j=1}^m f_jy_jR)\cap N$ and $k_0y_j=f_jy_jr_j\neq 0$ for some $r_j\in R$, $1\leqq j\leqq m$, also $k_{0|N}\in K(N)$ since $k_0(N)\subseteq N$. Now we let $r=\sum_{i=1}^m a_ir_i$. Then $x_ir=k_0y_i$ for all $1\leqq i\leqq n$.

2.7. PROPOSITION. *Let M be a quasi-simple R-module for some ring R. If the singular submodule of M is zero, then M is a quasi-simple B-module for any large right ideal B of R.*

PROOF. Let $f\in \mathrm{Hom}_B(N, M)$ for a submodule N of M. Suppose $f(mr_0)-f(m)r_0\neq 0$ for some $m\in N$ and $r_0\in R$. Let $A=\{a\in R \mid r_0a\in B\}$. Since B is a large right ideal, so is A. Let $G=A\cap B$. Then G is also a large right ideal of R and $(f(mr_0)-f(m)r_0)g=f(mr_0)g-f(m)r_0g =f(mr_0g)-f(mr_0g)=0$ for any $g\in G$. Since the singular submodule of M is zero, this implies that $f(mr_0)-f(m)r_0=0$, a contradiction. Thus $f\in \mathrm{Hom}_R(N, M)$ and $\mathrm{Hom}_R(N, M)\subseteq \mathrm{Hom}_B(N, M)\subseteq \mathrm{Hom}_R(N, M)$. Therefore $\mathrm{Hom}_R(N, M)=\mathrm{Hom}_B(N, M)$. Likewise, since the singular submodule of $\tilde{M}$ is zero, $\mathrm{Hom}_R(\tilde{M}, \tilde{M})=\mathrm{Hom}_B(\tilde{M}, \tilde{M})$. Now if N is a nonzero B-submodule of M, then $NB\subseteq N$ and $NB\neq 0$. Since NB is a nonzero R-submodule of M, there is $0\neq f\in \mathrm{Hom}_R(M, M) =\mathrm{Hom}_B(M, M)$ such that $f(M)\subseteq NB\subseteq N$. This proves that M is a quasi-simple B-module.

2.8. COROLLARY. *If the singular submodule of M in 2.6. is zero then, for any large right ideal B of R, one can choose $b\in B$ and $0\neq k \in \mathrm{Hom}_R(M, M)$ such that $k|_N\in K(N)$ and $x_ib=ky_i$ for all $1\leqq i\leqq n$.*

PROOF. This is a direct consequence of 2.6. and 2.7.

2.9. THEOREM. *Let R be a ring and M be a faithful finite dimensional quasi-simple R-module. Then every large right ideal B of R contains a regular element.*

PROOF. Let M_r^Δ be the singular submodule of M. We will show that $M_R^\Delta=0$. Suppose that $M_r^\Delta\neq 0$, then by 2.2 $\tilde{M}=DM_r^\Delta$ where $D =\mathrm{Hom}_R(\tilde{M}, \tilde{M})$. Since M is finite dimensional, there exists a finite basis $\{m_1, m_2, \cdots, m_n\}$ in M_r^Δ such that $\tilde{M}=\sum_{i=1}^n Dm_i$. It would follow that $\tilde{M}^r=\bigcap_{i=1}^n (m_i)^r\neq 0$ contradicting the faithfulness of M. Now, let $\{x_1, x_2, \cdots, x_n\}$ be a basis for $\tilde{M}$, where x_i can be chosen from M by 2.2. Then by 2.8, there are $b\in B$ and $0\neq k\in \mathrm{Hom}_R(M, M)$ such that $x_ib=kx_i$ for all $1\leqq i\leqq n$. Since $\{kx_1, kx_2, \cdots, kx_n\}$ is also a basis of $\tilde{M}$, b must be a regular element.

3.0. PROPOSITION. *If R is a prime right Goldie ring, then there is a faithful finite dimensional quasi-simple R-module M.*

PROOF. Take M to be any uniform right ideal of R. Then the regular R-module M is a faithful quasi-simple R-module as was proven in §1. It is well known that if $m\in M$ then $(m)^r$ is a closed right ideal. Suppose that M were not finite dimensional over D, where $D=\mathrm{Hom}_R(\tilde{M}, \tilde{M})$, and that $\{m_1, m_2, \cdots\}$ were an infinite D-independent set in M. Since any right Goldie ring satisfies the descending chain condition on closed right ideal (see for example [**5**, 3, p. 264]), the sequence $(m_1)^r\supset (m_1)^r\cap (m_2)^r\supset \cdots$, would terminate at finite number of places, say $\bigcap_{i=1}^{n} (m_i)^r = \bigcap_{i=1}^{n+1} (m_i)^r$. Thus, $(m_{n+1})^r \supseteq \bigcap_{i=1}^{n} (m_i)^r$ and $m_{n+1} \in Dm_1 + Dm_2 + \cdots + Dm_n$ by 2.5, a contradiction. Therefore M is finite dimensional.

If R is a prime ring with zero singular ideal and if R contains a uniform right ideal then the injective hull $\hat{R}$ of the regular R-module R is a prime regular ring with a minimal right ideal (refer to [**6**, 2.7 and 3.1]). Furthermore, if $a \in R$ such that $(a)^r = 0$ then there is $x \in \hat{R}$ such that $xa = 1$. We use these facts to prove the following theorem.

3.1. THEOREM. *Let R be a ring and M be an n-dimensional faithful quasi-simple R-module for some positive integer n. Let $D = \mathrm{Hom}_R(\tilde{M}, \tilde{M})$ and $\mathfrak{L} = \mathrm{Hom}_D(\tilde{M}, \tilde{M})$. Then the injective hull $\hat{R}$ of the regular R-module R is isomorphic (ring) to $\mathfrak{L}$.*

PROOF. First we note that by 2.0, $M^r = 0$ is a prime ideal of R, so R is a prime ring. According to 2.9, the singular ideal of R is zero. Let m be a nonzero element of M. Then there exists a nonzero right ideal U in R such that $(m)^r \cap U = 0$. Since every pair of nonzero submodules of M has a nonzero intersection, U must be a uniform right ideal of R. Thus $\hat{R}$ is a prime regular ring with a minimal right ideal. Let S be an arbitrary nonzero ideal in $\hat{R}$. Then $S \cap R$ is a nonzero ideal in R and it is a large right ideal of R. Hence, by 2.9, $S \cap R$ contains a regular element, say a. Therefore, there is $x \in \hat{R}$ such that $1 = xa \in xS \subseteq S$. Thus $S = \hat{R}$. This means that $\hat{R}$ is a simple ring with a minimal right ideal. Since $1 \in \hat{R}$, $\hat{R}$ is a simple artinian ring, and is isomorphic to the ring of all linear transformations of a finite dimensional vector space. Any $a \in \hat{R}$ such that either $ax = 1$ or $xa = 1$ for some $x \in R$ is a unit. Hence in particular any regular element of R is a unit in $\hat{R}$. Since if $q \in \hat{R}$ then the right ideal $(R:q) = \{r \in R \mid qr \in R\}$ is large, by 2.9 there is a regular element $a \in (R:q)$ such that $qa = b \in R$. Therefore $q = ba^{-1}$ and R is a right order in $\hat{R}$. Hence $\hat{R}$ is isomorphic to a subring of $\mathfrak{L}$ and $\tilde{M}$ is a faithful $\hat{R}$-module. Since every $\hat{R}$-module is injective (see [**1**, 4.2, p. 11]) and $\tilde{M}$ is a uniform $\hat{R}$-module, $\tilde{M}$ has no proper nonzero submodules. Note that $\mathrm{Hom}_{\hat{R}}(\tilde{M}, \tilde{M}) = \mathrm{Hom}_R(\tilde{M}, \tilde{M})$. Thus $\tilde{M}$ is an n-dimensional $\hat{R}$-module and $R \cong \mathfrak{L}$.

3.2. THEOREM. *A ring R has a faithful finite dimensional quasi-simple right module if and only if R is a right Goldie prime ring. When the conditions hold, a faithful right R-module M is finite dimensional and quasi-simple if and only if it is isomorphic to a uniform right ideal of R.*

PROOF. It is clear that every Goldie prime ring has a uniform right ideal which is a faithful finite dimensional quasi-simple right module.

Now we assume that R has a faithful finite dimensional quasi-simple right module, say M. Let $D=\mathrm{Hom}_R(\tilde{M}, \tilde{M})$ and $\mathfrak{L}=\mathrm{Hom}_D(\tilde{M}, \tilde{M})$. By the proof of 3.1, $\mathfrak{L}$ is a simple artinian ring and R is a right order in $\mathfrak{L}$. Hence R is a right Goldie prime ring. Moreover, since $\tilde{M}$ has no proper nonzero submodule, $\tilde{M}_{\mathfrak{L}}$ is isomorphic to a minimal right ideal of $\mathfrak{L}$. Identify $\tilde{M}$ with that right ideal. Since R is large in $\mathfrak{L}_R$, $M\cap R\neq 0$, and since $M\cap R\subseteq \tilde{M}$, we see that $M\cap R$ is a uniform right ideal of R. Finally, by the definition of quasi-simplicity, there is a nonzero $f\in D$ such that $f(M)\subseteq M\cap R$. Since all elements of D are monomorphisms, the right ideal $f(M)$ is the required isomorphic copy of M.

3.3. LEMMA. *Let D be a division ring and K be a right order in D. Then the $n\times n$ matrix ring K_n over K is a right order in D_n where D_n is the $n\times n$ matrix ring over D and, for any nonsingular matrix $(k_{ij})\in K_n$, there exists a scalar matrix $\bar{k}$ which is determined by some $k\in K$ and a matrix $(k'_{ij})\in K_n$ such that $(k_{ij})^{-1}=(k\bar{k}'_{ij})^{-1}$.*

PROOF. See [8, p. 114].

3.4. THEOREM. *Let R be a ring and M be an n-dimensional faithful quasi-simple R-module for some positive integer n. Let $D=\mathrm{Hom}_R(\tilde{M}, \tilde{M})$ and $K(N)=\mathrm{Hom}_R(N, N)$, where N is a nonzero submodule of M. Let R^* be the subring of D_n which represents R as a ring of linear transformations on $\tilde{M}$ over D relative to the basis on which D_n represents $\mathrm{Hom}_D(\tilde{M}, \tilde{M})$. Then $K(N)_n\cap R^*$ is a right order in D_n and $[K(N)]\cap R^*$ is a right order in $\bar{D}$ where $[K(N)]$, $\bar{D}$ are the rings of scalar matrices over $K(N)$ and D respectively.*

PROOF. By 2.2, $\tilde{M}=DN$. Let $\{m_1, m_2, \cdots, m_n\}\subseteq N$ be a basis of $\tilde{M}$ on which D_n represents $\mathfrak{L}=\mathrm{Hom}_D(\tilde{M}, \tilde{M})$. If $q\in D_n$ then $q=(k_{ij})\bar{k}^{-1}$ for some $(k_{ij})\in K(N)_n$ and $\bar{k}\in[K(N)]$ by 3.2. By 3.1, $q=r_1r_2^{-1}$ for some $r_1, r_2\in R^*$. Since $\bar{k}^{-1}r_2$ is regular, the set $\{m_1\bar{k}^{-1}r_2, m_2\bar{k}^{-1}r_2, \cdots, m_n\bar{k}^{-1}r_2\}$ is linearly independent. Hence by 2.6 there exist $r\in R^*$ and $0\neq k_0\in\mathrm{Hom}_R(M, M)$ such that $m_i\bar{k}^{-1}r_2r=k_0m_i=m_i\bar{k}_0$ for all $1\leqq i\leqq n$ and $k_{0|N}\in K(N)$. This means that $\bar{k}^{-1}r_2r=\bar{k}_0$ and $r_2r=\bar{k}\bar{k}_0\in[K(N)]\cap R^*$. Now $r_1r=(k_{ij})\bar{k}^{-1}r_2r=(k_{ij})\bar{k}_0\in K(N)_n\cap R^*$. Therefore $q=(k_{ij})\bar{k}^{-1}=(k_{ij})\bar{k}_0\bar{k}_0^{-1}\bar{k}^{-1}=(r_1r)(r_2r)^{-1}$ and $K(N)_n\cap R^*$ is a right order in D_n. To see that $[K(N)]\cap R^*$ is a right order in $\bar{D}$, let $\bar{d}\in\bar{D}$. Then $\bar{d}=(k_{ij})\bar{k}^{-1}$ for some $(k_{ij})\in K(N)_n\cap R^*$ and $\bar{k}\in[K(N)]\cap R^*$ by the above argument. Therefore $\bar{d}\bar{k}=(k_{ij})$. Now $m_i\bar{d}\bar{k}=m_i(k_{ij})$, for all $1\leqq i\leqq n$, so

$$k_{ij} = dk \quad \text{if } i=j,$$
$$= 0 \quad \text{otherwise.}$$

This means that $(k_{ij}) \in [K(N)] \cap R^*$ and $[K(N)] \cap R^*$ is a right order in $\overline{D}$.

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