THE ANNIHILATOR OF RADICAL POWERS IN THE MODULAR GROUP RING OF A p-GROUP

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ABSTRACT. We show that if N is the radical of the group ring and L is the exponent of N, then the annihilator of N^w is N^{L-w+1}. As corollaries we show that the group ring has exactly one ideal of dimension one and if the group is cyclic, then the group ring has exactly one ideal of each dimension.

This paper deals with the group ring of a group of prime power order over the field of integers modulo p, where p is the prime dividing the order of the group. This field is written as K and the group ring as KG. It is well known that KG is not semisimple; if N is the radical of KG and $N^L \neq 0$ while $N^{L+1}=0$, then L is said to be the exponent of N. We prove the following result:

THEOREM. *Let G be a p-group and KG be the group ring of G over $K=GF(p)$, the field with p elements. If L is the exponent of the radical, N, of KG, then the annihilator of N^w is N^{L-w+1}.*

For S a nonempty subset of G, let $S^+=\sum_{g_i \in S} g_i$; in particular, for H a normal subgroup of G, let $\langle H^+\rangle$ be the ideal in KG generated by H^+. For g and h in G, the following identities are used:

$$(g-1)^{p-1} = 1 + g + g^2 + \cdots + g^{p-1}; \quad (g-1)^p = g^p - 1;$$
$$(gh-1) = (g-1)(h-1) + (g-1) + (h-1);$$

and

$$(h-1)(g-1) = (g-1)(h-1) + (gh-1)(c-1) + (c-1)$$

where $c=(h,g)=h^{-1}g^{-1}hg$. The following definitions and theorems are due to Jennings [1].

Let K_i be the set of all elements g_i in G such that $g_i \equiv 1 \bmod N^i$.

THEOREM 1 [1, THEOREM 2.2]. *The sets K_i, $i=1, 2, \cdots$, form a decreasing series of characteristic subgroups of G.*

This series of subgroups will be referred to as the K-series of G.

THEOREM 2 [1, THEOREM 2.3]. *The K-series of any group G has the following properties:*

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(1) $(K_i, K_j) \subseteq K_{i+j}$;
(2) *if* g_i *is in* K_i, *then* g_i^p *is in* K_{ip}; *and*
(3) K_i/K_{2i} *is elementary abelian.*

By Theorem 2, K_i/K_{i+1} is elementary abelian. Let K_i/K_{i+1} have order p^{d_i} ($d_i = 0$ if $K_i = K_{i+1}$), and let $g_{i,1}, \cdots, g_{i,d_i}$ be a complete set of representatives in G of a minimal basis for K_i/K_{i+1} (if $K_i = K_{i+1}$, let $g_{i,j} = 1$). In terms of these $g_{i,j}$, for fixed w, consider all products of the form $\Pi_{i,j}(g_{i,j}-1)^{a_{i,j}}$, for $0 \leqq a_{i,j} < p$, with $\sum_{i,j}(ia_{i,j}) = w$; the summation extending over the same i and j as in the product. Those factors which are present in the product are in the natural order of increasing i and j. Call these various distinct products $N_1^w, N_2^w, \cdots, N_{t_w}^w$.

THEOREM 3 [**1**, THEOREM 3.6]. *The elements* N_i^w *for fixed* w *form a basis for* N^w *modulo* N^{w+1}. *The number* t_w *of these elements is the dimension of* N^w/N^{w+1}.

THEOREM 4 [**1**, THEOREM 3.7]. *The dimension* t_w *of* N^w/N^{w+1} *is equal to the coefficient of* x^w *in the expansion of*

$$(1 + x + x^2 + \cdots + x^{p-1})^{d_1} \cdots (1 + x^i + x^{2i} + \cdots + x^{i(p-1)})^{d_i} \cdots$$

and the exponent L *of* N *is equal to* $\sum_i id_i(p-1)$.

The M-series for G is defined as follows: $M_1 = G$; for $i > 1$, $M_i = \langle (M_{i-1}, G), M^p_{(i/p)} \rangle$ where (i/p) is the least integer not greater than i/p and M_k^p is the set of all pth powers of elements of M_k.

THEOREM 5 [**1**, THEOREM 5.5]. *The* M-*series and the* K-*series are identical.*

The notation, definitions, and results of Jennings stated above will be used throughout this paper.

LEMMA 6. *If* $G_1 \supset G_2 \supset \cdots \supset G_m = 1$ *is a composition series in* G *and* g_i *is a representative in* G *of a minimal basis for* G_i/G_{i+1}, *then* $(g_i - 1)^{p-1}G_{1+1}^+ = G_i^+$.

PROOF. The cosets of G_i/G_{i+1} will be $G_{i+1}, g_iG_{i+1}, \cdots, g_i^{p-1}G_{i+1}$. Hence, $(g_i-1)^{p-1}G_{i+1}^+ = (1 + g_i + g_i^2 + \cdots + g_i^{p-1})G_{i+1}^+ = G_i^+$.

THEOREM 7. *Let* $x = \prod_{i \geqq m}(g_{i,j}-1)^{p-1}$ *for some fixed positive integer* m, *the factors being in the natural order of increasing* i *and* j, *and the product taken over all* $(g_{i,j}-1)$ *with* $d_i \neq 0$. *If* m *is such that* $d_j \neq 0$ *for some* $j \geqq m$, *then* $x = M_m^+$.

PROOF. The proof is inductive. Let k be the largest integer such that $d_k \neq 0$. By Theorems 2 and 5, M_k is elementary abelian. Notice that $M_k = \langle g_{k,1}, g_{k,2}, \cdots, g_{k,d_k}\rangle \supset \langle g_{k,2}, \cdots, g_{k,d_k}\rangle \supset \cdots \supset \langle g_{k,d_k}\rangle \supset 1$ is a composition series, so that Lemma 6 applies and $M_k^+ = \prod (g_{k,j}-1)^{p-1}$. Hence the theorem is proved for $m=k$.

Suppose the theorem is true for $m=r+1$. If $d_r=0$, then $M_r=M_{r+1}$ and there is nothing to prove. If $d_r \neq 0$, then $x = \prod_{i \geqq r} (g_{i,j}-1)^{p-1} = \prod (g_{r,j}-1) M_{r+1}^+$. By Theorems 2 and 5; M_r/M_{r+1} is elementary abelian and, as above, $x = M_r^+$. Hence the theorem is true for all applicable m.

In particular, x of this form for $m=1$ implies that $x = M_1^+ = G^+$. Therefore, $N^L = \langle G^+ \rangle$.

Define a function f on the N_k^w, $f(N_k^w)$ a product of powers of the $(g_{i,j}-1)$ such that: if $d_i=0$, then the term $(g_{i,j}-1)$ does not appear in $f(N_k^w)$; if $d_i \neq 0$, then $f(N_k^w)$ has a factor $(g_{i,j}-1)^{b_{i,j}}$ with $b_{i,j} = (p-1)-a_{i,j}$ where $a_{i,j}$ is the exponent on the corresponding term in N_k^w. The factors are arranged in the natural order of increasing i and j.

Notice that $0 \leqq b_{i,j} < p$ and the factors are in the proper order so that $f(N_k^w)$ is some N_r^m, a basis element for N^m/N^{m+1}. Since $m = \sum_{i,j} ib_{i,j} = \sum_{i,j} i((p-1)-a_{i,j})$ and for each i summing over j yields d_j factors of $(p-1)$, $m = L-w$ by Theorem 4. Hence, $f(N_k^w)$ is some N_r^{L-w}. Clearly, $f(f(N_k^w)) = N_k^w$.

LEMMA 8. *For f defined above, $f(N_k^w) \cdot N_k^w = G^+$.*

PROOF. Let t be an integer such that $M_t=1$. Then for $m=t$ the product has the form

$$f(N_k^w) \cdot N_k^w = \pi_1 \pi_2 M_m^+ \tag{1}$$

where π_1 is a product of $(g_{i,j}-1)$ from $f(N_k^w)$ with $i<m$ and π_2 is a similar product from N_k^w.

Suppose the product has the form (1) for $m=r+1$. If $d_r=0$, then $M_r=M_{r+1}$ and the product has form (1) for $m=r$. If $d_r \neq 0$, write

$$f(N_k^w) \cdot N_k^w = \pi_1' (g_{r,j}-1)(g_{s,i}-1)\pi_2' M_{r+1}^+$$

and let $c = (g_{r,j}, g_{s,i})$. Then

$$(g_{r,j}-1)(g_{s,i}-1) = (g_{s,i}-1)(g_{r,j}-1) + (g_{s,i}g_{r,j})(c-1),$$

so that

$$\begin{aligned} f(N_j^w) \cdot N_j^w = {} & \pi_1' (g_{s,i}-1)(g_{r,j}-1)\pi_2' M_{r+1}^+ \\ & + \pi_1' (g_{s,i}g_{r,j})(c-1)\pi_2' M_{r+1}^+. \end{aligned}$$

By Theorems 2 and 5, c is in M_{r+1}; the second term in the sum has factors of $(c-1)$ and M_{r+1}^{+}, so this term is 0. Hence, the $(g_{r,j}-1)$ terms commute to their natural position in the product. Since the exponent on the $(g_{r,j}-1)$ term in the product is $a_{r,j}+b_{r,j}=p-1$, the product has form (1) for $m=r$ by Theorem 7. Therefore, the product has form (1) for $m=1$ and the lemma is proved.

Let $a_{i,j}$ be the exponent on the $(g_{i,j}-1)$ term in N_k^w and let $b_{i,j}$ be the exponent on this term in N_m^w; if the $(g_{i,j}-1)$ term does not occur in N_k^w, let $a_{i,j}=0$ with similar convention in N_m^w. Define $N_k^w \ll N_m^w$ if there are positive integers s and t such that $a_{s,t}<b_{s,t}$ and $a_{i,j}=b_{i,j}$ for all i and j such that $i>s$ or $i=s$ and $j>t$. That is, terms to the right of $(g_{s,t}-1)$ have the same exponents and $a_{s,t}<b_{s,t}$. Clearly any two distinct basis elements are comparable and this ordering is transitive.

LEMMA 9. *If* $f(N_k^w) \ll N_m^{L-w}$, *then* $N_m^{L-w}\cdot N_k^w=0$.

PROOF. Let s and t be as in the definition of $f(N_k^w)\ll N_m^{L-w}$. As in Lemma 8, the terms $(g_{i,j}-1)$ with $i\geqq s+1$ commute to their natural position so the product has the form

$$N_m^{L-w}\cdot N_k^w = \pi_1\pi_2 M_{s+1}^{+}.$$

The terms $(g_{s,j}-1)$ also commute to their natural position by the proof of Lemma 8. The exponent on the $(g_{s,t}-1)$ term in the product is $a_{s,t}+b_{s,t}$ where $a_{s,t}$ is the exponent from the term in N_m^{L-w} and $b_{s,t}$ is the exponent in N_w^k. The exponent of $(g_{s,t}-1)$ in $f(N_k^w)$ is $(p-1)-b_{s,t}$ and $a_{s,t}>(p-1)-b_{s,t}$ so that $a_{s,t}+b_{s,t}>(p-1)$. Let $a_{s,t}+b_{s,t}=p+c_{s,t}$. Then $(g_{s,t}-1)^{a_{s,t}+b_{s,t}}=(g_{s,t}^p-1)(g_{s,t}-1)^{c_{s,t}}$. By Theorem 2, $g_{s,t}^p$ is in M_{s+1}. The product has factors of $(g_{s,t}^p-1)$ and M_{s+1}^{+}; therefore, the product is 0 and the lemma is proved.

For I an ideal in KG, let $0:I$ be the left annihilator of I and $A(I)$ be the annihilator (two-sided) of I.

LEMMA 10. $0:N^w=N^{L-w+1}$.

PROOF. Clearly $0:N^L=N$; suppose that $0:N^{w+1}=N^{L-w}$. Then $N^{w+1}\subseteq N^w$ implies $0:N^w\subseteq 0:N^{w+1}=N^{L-w}$ so that $N^{L-w+1}\subseteq 0:N^w \subseteq N^{L-w}$. The lemma will be proved if x is a linear combination of the N_i^{L-w} and x is in $0:N^w$ implies $x=0$.

For f defined above, if $N_k^w\ll N_m^w$, then clearly $f(N_m^w)\ll f(N_k^w)$. Let N_1^w be the least N_i^w under $\ll$, N_2^w the second smallest and so on; let N_m^w be the largest. Order the N_i^{L-w} in the same manner. Then $f(N_m^w)=N_1^{L-w}$ and for $m>k\geqq 0$, $f(N_{m-k}^w)=N_{k+1}^{L-w}$ since f reverses the order.

Let $x=\sum a_i N_i^{L-w}$ be a linear combination of the N_i^{L-w} ordered as above and suppose that x is in $0:N^w$. By Lemmas 8 and 9, $x\cdot N_m^w=a_1G^+$ since $N_1^{L-w}\cdot N_m^w=f(N_m^w)\cdot N_m^w=G^+$ and $N_i^{L-w}\gg f(N_m^w)$ implies $N_i^{L-w}\cdot N_m^w=0$. Hence, if x is in $0:N^w$, then $a_1=0$. Suppose that x in $0:N^w$ implies $a_i=0$ for $i\leqq k$. As above, $x\cdot N_{m-k}^w=a_{k+1}G^+$ since $a_i=0$ for $i\leqq k$, $N_{k+1}^{L-w}\cdot N_{m-k}^w=f(N_{m-k}^w)\cdot N_{m-k}^w=G^+$, and $N_i^{L-w}\cdot N_{m-k}^w=0$ for $N_i^{L-w}\gg f(N_{m-k}^w)$. Therefore, x in $0:N^w$ implies $a_{k+1}=0$ so that $a_i=0$ for $1\leqq i\leqq m$ and $x=0$.

The proof of the main theorem follows immediately.

THEOREM 11. $A(N^w)=N^{L-w+1}$.

PROOF. By Lemma 10, $N^{L-w+1}=0:N^w\supseteq A(N^w)\supseteq N^{L-w+1}$.

COROLLARY 12. *If I is an ideal of dimension w and $w\leqq L$, then $I\subseteq N^{L-w+1}$.*

PROOF. For I as above. $I\cdot N^w=N^w\cdot I=0$ so that $I\subseteq A(N^w)=N^{L-w+1}$.

COROLLARY 13. *KG has exactly one ideal of dimension one; this ideal is $\langle G^+\rangle$.*

PROOF. If I has dimension one, then $I\subseteq N^L=\langle G^+\rangle$; since $\langle G^+\rangle$ has dimension one, $I=\langle G^+\rangle$.

COROLLARY 14. *G cyclic implies KG has exactly one ideal of each dimension.*

PROOF. If G is cyclic of order p^n, then $N^{p^n}=0$ and $N^{p^n-1}\neq 0$. Hence $(g-1)^w$ is a basis for N^w/N^{w+1} where g generates G. Also, N^{L-w+1} has dimension w. If I is an ideal of dimension w, then $I\cdot(g-1)^w=0$; therefore, $I\subseteq 0:N^w=N^{L-w+1}$. Since I and N^{L-w+1} have the same dimension, they are equal.

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