A NOTE ON SUBPARACOMPACT SPACES

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ABSTRACT. The main result in this paper states that every metacompact space in which every closed set is a G_δ is subparacompact. The class of countably subparacompact spaces is introduced and several results about such spaces are proved.

1. **Introduction.** A topological space is said to be *subparacompact* if every open cover has a σ-discrete closed refinement. This class of spaces was introduced by McAuley in [8] (he called such spaces *F_σ-screenable*) and more recently has been studied by Burke [2] and Creede [3]. Every paracompact space is subparacompact, and Creede proved that every semistratifiable space is subparacompact. The main result of this paper is the following theorem.

THEOREM 1. *Every metacompact space in which every closed set is a G_δ is subparacompact.*

In connection with this result it should be pointed out that Worrell [9] has given an example of a metacompact space which fails to be subparacompact.

In §2 we give the proof of Theorem 1. In §3 we introduce the class of countably subparacompact spaces and prove several results about such spaces. Finally, in §4 we summarize the theorems in this paper together with some results from the literature.

A topological space is *metacompact* if every open cover has a point finite open refinement. According to Bing [1] a topological space is *screenable* if every open cover has a σ-disjoint open refinement. No separation axioms are assumed in this paper.

2. **Proof of Theorem 1.** The following notation will be used in this section: given a set X, a point p in X, and a cover $\mathcal{U}$ of X, $\operatorname{ord}(p, \mathcal{U})$ is the number of elements of $\mathcal{U}$ containing p. Theorem 1 follows immediately from this Lemma.

LEMMA. *Let X be a topological space in which every closed set is a G_δ. Then every point finite open cover of X has a σ-discrete closed refinement.*

PROOF. Let $\mathcal{U} = \{U_\alpha\colon \alpha \text{ in } A\}$ be a point finite open cover of X,

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let $\leqq$ be a well ordering for A. We shall construct a σ-discrete closed refinement $\mathcal{H}$ of $\mathcal{U}$. For $n=1, 2, \cdots$ let $F_n = \{x \text{ in } X: \text{ord}(x, \mathcal{U}) \leqq n\}$. It is easy to check that F_n is a closed set and so $F_n = \bigcap_{i=1}^{\infty} W_{ni}$, where each W_{ni} is an open set. Since each U_α is an open set we have $U_\alpha = \bigcup_{j=1}^{\infty} K_{\alpha j}$, where each $K_{\alpha j}$ is a closed set. For $n=1, 2, \cdots$, $i=1, 2, \cdots$, $j=1, 2, \cdots$, and α in A let

$$H_{\alpha nij} = \left(K_{\alpha j} - \bigcup_{\beta<\alpha} U_\beta\right) \cap (F_n - W_{(n-1)i}) \qquad (W_{0i} = \varnothing),$$

let $\mathcal{H}_{nij} = \{H_{\alpha nij}: \alpha \text{ in } A\}$, and let $\mathcal{H} = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \mathcal{H}_{nij}$.

Clearly each element of $\mathcal{H}$ is closed and is contained in some element of $\mathcal{U}$. To see that $\mathcal{H}$ covers X, consider p in X. Let α be the smallest element of A such that p belongs to U_α, and let $\text{ord}(p, \mathcal{U}) = n$. Then there is some i such that p is not in $W_{(n-1)i}$ and there is some j such that p is in $K_{\alpha j}$. Thus p is in $H_{\alpha nij}$.

It remains to show that each $\mathcal{H}_{nij}$ is a discrete collection. Let p be an arbitrary point of X. If $\text{ord}(p, \mathcal{U}) > n$ then p has a neighborhood which misses F_n and so p has a neighborhood which misses all elements of $\mathcal{H}_{nij}$. If $\text{ord}(p, \mathcal{U}) < n$ then $W_{(n-1)i}$ is a neighborhood of p which misses all elements of $\mathcal{H}_{nij}$. Finally suppose $\text{ord}(p, \mathcal{U}) = n$ with p belonging to $U_{\alpha_1}, \cdots, U_{\alpha_n}$, $\alpha_1 < \cdots < \alpha_n$, let $V = \bigcap_{i=1}^{n} U_{\alpha_i}$, and consider α in A. If $\alpha < \alpha_1$ then $V \cap H_{\alpha nij} = \varnothing$ since x in $V \cap H_{\alpha nij}$ implies $\text{ord}(x, \mathcal{U}) > n$ and x in F_n, a contradiction. If $\alpha > \alpha_1$ then $U_{\alpha_1} \cap H_{\alpha nij} = \varnothing$ and so $V \cap H_{\alpha nij} = \varnothing$. Thus V is a neighborhood of p which intersects at most one element of $\mathcal{H}_{nij}$, namely $H_{\alpha_1 nij}$.

3. **Countably subparacompact spaces.** A topological space is said to be *countably subparacompact* if every countable open cover has a σ-discrete closed refinement. This concept was studied by Mansfield in [7], where he proved that in normal spaces countable subparacompactness is equivalent to countable paracompactness.

It is easy to show that a topological space X is countably subparacompact if and only if it satisfies this condition: given a countable open cover $\{U_n: n=1, 2, \cdots\}$ of X, there is a countable closed cover $\{F_{nj}: n=1, 2, \cdots, j=1, 2, \cdots\}$ of X with $F_{nj} \subseteq U_n$ for all n and j. Theorem 2 below follows immediately from this characterization of countable subparacompactness.

THEOREM 2. *Every topological space in which every closed set is a G_δ is countably subparacompact.*

The relation between countable metacompactness and countable subparacompactness is as follows.

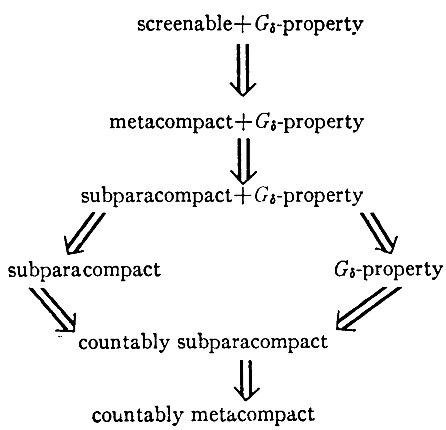
THEOREM 3. *Every countably subparacompact space is countably metacompact.*

PROOF. Let $\{U_n: n=1, 2, \cdots\}$ be a countable open cover of X. It suffices to construct a point finite open cover $\{V_n: n=1, 2 \cdots\}$ of X with $V_n \subseteq U_n$ for all n. Let $\{F_{nj}: n=1, 2, \cdots, j=1, 2, \cdots\}$ be a closed cover of X with $F_{nj} \subseteq U_n$ for all n and j. Let $V_1 = U_1$ and for $n=2, 3, \cdots$ let $V_n = U_n - \bigcup\{F_{kj}: k<n, j<n\}$. It is easy to check that $\{V_n: n=1, 2, \cdots\}$ is the desired point finite open cover of X.

EXAMPLE. *A countably metacompact space not countably subparacompact.* In [2] Burke gives an example of a metacompact space X which fails to be subparacompact. It is easy to check that the space X is also screenable. It follows from Theorem 4 below that X cannot be countably subparacompact.

THEOREM 4. *Every screenable countably subparacompact space is subparacompact.*

PROOF. Let $\mathcal{W}$ be an open cover of X. We want to construct a σ-discrete closed refinement $\mathcal{H}$ of $\mathcal{W}$. Let $\bigcup_{n=1}^{\infty} \mathcal{U}_n$ be a σ-disjoint open refinement of $\mathcal{W}$, and for $n=1, 2, \cdots$ let $U_n = \bigcup \mathcal{U}_n$. Then $\{U_n: n=1, 2, \cdots\}$ is a countable open cover of X so there is a closed cover $\{F_{nj}: n=1, 2, \cdots, j=1, 2, \cdots\}$ of X with $F_{nj} \subseteq U_n$ for all n and j. Let $\mathcal{H}_{nj} = \{U \cap F_{nj}: U \in \mathcal{U}_n\}$ and let $\mathcal{H} = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\infty} \mathcal{H}_{nj}$. Clearly $\mathcal{H}$ covers X and each element of $\mathcal{H}$ is contained in some element of $\mathcal{W}$. Each element of $\mathcal{H}$ is closed since $F_{nj} \cap U = F_{nj} \cap (X - \bigcup\{U': U' \in \mathcal{U}_n, U' \neq U\})$, and it is easy to verify that each $\mathcal{H}_{nj}$ is discrete. Thus $\mathcal{H}$ is the desired σ-discrete closed refinement of $\mathcal{W}$.



4. **Summary.** For convenience we shall say that a topological space has the *G_δ-property* if every closed set is a G_δ. Heath [6] proved that every screenable space with the G_δ-property is metacompact, and Greever [4], [5] generalized this result by showing that every space with the G_δ-property is countably metacompact and that every screenable countably metacompact space is metacompact. These results, together with the theorems in this paper, can be summarized in the preceding diagram.

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