APPROXIMATING RESIDUAL SETS BY STRONGLY RESIDUAL SETS[1]

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ABSTRACT. Let M be a closed topological manifold, R residual in M, and N any neighborhood of R in M. The fulfillment by R of a certain local separation property in M implies that there exists a topological spine R' of M such that $N \supset R' \supset R$. (Topological spine =strongly residual set.) This local separation property is satisfied whenever R is an ANR, or when $\dim R \leqq \dim M - 2$.

Let M be a closed topological n-manifold. Following Doyle and Hocking [2], a subset R of M is said to be *residual* in M if $M-R$ is a topological open n-cell which is dense in M. In [3], such a subset was called *strongly residual* if it can be realized as $\phi(\dot{I}^n)$ for some map ϕ satisfying the criteria of the mapping theorem of Brown and Casler [1]. The main result contained herein represents some progress toward determining which residual sets are strongly residual: it shows that a residual set possessing a certain property akin to semi-local-connectedness can be enlarged by an arbitrarily small amount to form a set which is strongly residual. The theorem is arrived at by tampering with the proof of the Brown-Casler theorem, and only the modifications will be given here. The definitions of the terms used below are by now standard, or may be found in [1].

THEOREM. *Let R be residual in M, and let N be any neighborhood of R in M. Suppose that there is a sequence $\mathcal{E}_1, \mathcal{E}_2, \cdots$ of finite open covers of M such that* (1) *the diameters* (*relative to some fixed metric on M*) *of the members of $\mathcal{E}_i$ tend to zero as $i \to \infty$, and* (2) *for each element E of any cover in the sequence, $E-R$ has finitely many components. Then there is a map ϕ from I^n onto M such that $\phi | \mathring{I}^n$ is a homeomorphism, $\phi^{-1}\phi(\dot{I}^n) = \dot{I}^n$, $\dim \phi(\dot{I}^n) \leqq n-1$, and $N \supset \phi(\dot{I}^n) \supset R$.*

PROOF. By using the Lebesgue numbers of the coverings $\mathcal{E}_i$ and relabeling a subsequence if necessary, it can easily be shown that no generality is lost by assuming that each element of $\mathcal{E}_i$ is a connected

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open set of diameter less than 2^{-i-1}, and that for each i, $\mathcal{E}_{i+1}$ refines $\mathcal{E}_i$.

Let C_1 be an n-cell in $M-R$ which contains a point of each component of $E-R$, for each element E of $\mathcal{E}_1$; C_1 can (and will) be chosen so that $M-C_1\subset N$, and $\dot{C}_1$ is bicollared. Let X_2 be a finite set of points of $M-R$ consisting of one point from each component of $E-R$, for each $E\in\mathcal{E}_2$. If $x\in X_2$, let E_2 be a member of $\mathcal{E}_2$ containing x, and suppose that K_2 is the component of E_2-R which contains x. Because K_2 is connected and $\mathcal{E}_2$ refines $\mathcal{E}_1$, K_2 lies in a component K_1 of E_1-R, for some $E_1\in\mathcal{E}_1$.

Since R is closed in M and E_1 is open in M, K_1 is a component of the open subset E_1-R of the locally-connected space M, hence is open in M. K_1 is contained the open n-cell $M-R$ and is thus open in $M-R$.

Now connectedness is equivalent to polygonal-path-connectedness for open subsets of euclidean spaces, so there is a polygonal arc (relative to some combinatorial structure on $M-R$) joining x to some point of $\dot{C}_1\cap E_1$, and lying entirely within K_1.

Mutually disjoint n-cells containing the appropriate subarcs of the polygonal arcs thus described can now be found, and an autohomeomorphism h_1 of M constructed with $h_1(C_1)$ engulfing X_2, exactly as in [1]. Repetition of the above process results in the definition of the map ϕ, the details of the iterative process being precisely parallel to the proof in [1], the only modifications being those analogous to those described above.

COROLLARY. *Let R be residual in M, and let N be any neighborhood of R in M. If* dim $R\leqq n-2$, *or if R is an ANR, then there exists R' strongly residual in M with $N\supset R'\supset R$.*

PROOF. If dim $R\leqq n-2$, any sequence of finite open cell covers of M whose members have diameters tending to zero will satisfy the hypotheses of the theorem, since an n-cell cannot be separated by a set of dimension $\leqq n-2$.

If R is an ANR, a sequence of covers of M satisfying the hypotheses of the theorem is easily constructed by using the following

LEMMA. *If X is an ANR closed in E^n,* dim $X\leqq n-1$, $x\in X$, *and V is any neighborhood of x, then there exists a neighborhood U of x such that $U\subset V$, and $U-X$ has finitely many components.*

PROOF. Let W be any compact neighborhood of x which is contained in V. By the last-stated lemma in [4], W meets only finitely many components of $V-X$. Let K be the union of these components, and put $U=\text{Int } \overline{K}$. U clearly has the required properties.

A residual set satisfying the hypotheses of the above theorem need not itself be strongly residual, even if it has dimension $n-1$, as the following example shows:

EXAMPLE. Consider S^3 as the one-point compactification of E^3, and define subsets of S^3 *via* a rectangular coordinate system in E^3, as follows:

$$A = \{(x, y, z) \mid z = 0, -1 \leqq x \leqq 0, -1 \leqq y \leqq 1\},$$
$$B = \{(x, y, z) \mid 0 < x \leqq 1, y = \sin(1/x), 0 \leqq z \leqq x\},$$
$$C = \{(x, y, z) \mid z = 0, 0 \leqq x \leqq \epsilon, -1 \leqq y \leqq 1\},$$

where ϵ is any small positive number.

$A \cup B$ is not strongly residual in S^3 since, failing to be locally connected, it cannot be an ANR (see [3]). However, $A \cup B$ can easily be shown to be cellular, and this implies that it is residual in S^3. Including the additional set C cleans matters up quite a bit; albeit tedious, it is by no means a herculean task to exhibit a pseudo-isotopy of the cube $\{(x, y, z) \mid -1 \leqq x, y, z \leqq 1\}$ onto $A \cup B \cup C$.

The question as to whether a residual ANR can fail to be strongly residual remains open.

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