APPROXIMATING RESIDUAL SETS BY STRONGLY RESIDUAL SETS¹

D. A. MORAN

Abstract. Let $M$ be a closed topological manifold, $R$ residual in $M$, and $N$ any neighborhood of $R$ in $M$. The fulfillment by $R$ of a certain local separation property in $M$ implies that there exists a topological spine $R'$ of $M$ such that $N \supseteq R' \supseteq R$. (Topological spine = strongly residual set.) This local separation property is satisfied whenever $R$ is an ANR, or when $\dim R \leq \dim M - 2$.

Let $M$ be a closed topological $n$-manifold. Following Doyle and Hocking [2], a subset $R$ of $M$ is said to be residual in $M$ if $M - R$ is a topological open $n$-cell which is dense in $M$. In [3], such a subset was called strongly residual if it can be realized as $\phi(I^n)$ for some map $\phi$ satisfying the criteria of the mapping theorem of Brown and Casler [1]. The main result contained herein represents some progress toward determining which residual sets are strongly residual: it shows that a residual set possessing a certain property akin to semi-local-connectedness can be enlarged by an arbitrarily small amount to form a set which is strongly residual. The theorem is arrived at by tampering with the proof of the Brown-Casler theorem, and only the modifications will be given here. The definitions of the terms used below are by now standard, or may be found in [1].

Theorem. Let $R$ be residual in $M$, and let $N$ be any neighborhood of $R$ in $M$. Suppose that there is a sequence $\mathcal{E}_1, \mathcal{E}_2, \ldots$ of finite open covers of $M$ such that (1) the diameters (relative to some fixed metric on $M$) of the members of $\mathcal{E}_i$ tend to zero as $i \to \infty$, and (2) for each element $E$ of any cover in the sequence, $E - R$ has finitely many components. Then there is a map $\phi$ from $I^n$ onto $M$ such that $\phi| I^n$ is a homeomorphism, $\phi^{-1}\phi(I^n) = I^n$, $\dim \phi(I^n) \leq n - 1$, and $N \supseteq \phi(I^n) \supseteq R$.

Proof. By using the Lebesgue numbers of the coverings $\mathcal{E}_i$, and relabeling a subsequence if necessary, it can easily be shown that no generality is lost by assuming that each element of $\mathcal{E}_i$ is a connected
open set of diameter less than $2^{-i-1}$, and that for each $i$, $\varepsilon_{i+1}$ refines $\varepsilon_i$.

Let $C_1$ be an $n$-cell in $M - R$ which contains a point of each component of $E - R$, for each element $E$ of $\varepsilon_1$; $C_1$ can (and will) be chosen so that $M - C_1 \subseteq N$, and $C_1$ is bicollared. Let $X_2$ be a finite set of points of $M - R$ consisting of one point from each component of $E - R$, for each $E \in \varepsilon_2$. If $x \in X_2$, let $E_2$ be a member of $\varepsilon_2$ containing $x$, and suppose that $K_2$ is the component of $E_2 - R$ which contains $x$. Because $K_2$ is connected and $\varepsilon_2$ refines $\varepsilon_1$, $K_2$ lies in a component $K_1$ of $E_1 - R$, for some $E_1 \in \varepsilon_1$.

Since $R$ is closed in $M$ and $E_1$ is open in $M$, $K_1$ is a component of the open subset $E_1 - R$ of the locally-connected space $M$, hence is open in $M$. $K_1$ is contained the open re-cell $M - R$ and is thus open in $M - R$.

Now connectedness is equivalent to polygonal-path-connectedness for open subsets of euclidean spaces, so there is a polygonal arc (relative to some combinatorial structure on $M - R$) joining $x$ to some point of $C_1 \cap E_1$, and lying entirely within $K_1$.

Mutually disjoint re-cells containing the appropriate subarcs of the polygonal arcs thus described can now be found, and an autohomeomorphism $h_1$ of $M$ constructed with $h_1(C_1)$ engulfing $X_2$, exactly as in [1]. Repetition of the above process results in the definition of the map $\phi$, the details of the iterative process being precisely parallel to the proof in [1], the only modifications being those analogous to those described above.

**Corollary.** Let $R$ be residual in $M$, and let $N$ be any neighborhood of $R$ in $M$. If $\dim R \leq n - 2$, or if $R$ is an ANR, then there exists $R'$ strongly residual in $M$ with $N \subseteq R' \subseteq R$.

**Proof.** If $\dim R \leq n - 2$, any sequence of finite open cell covers of $M$ whose members have diameters tending to zero will satisfy the hypotheses of the theorem, since an $n$-cell cannot be separated by a set of dimension $\leq n - 2$.

If $R$ is an ANR, a sequence of covers of $M$ satisfying the hypotheses of the theorem is easily constructed by using the following

**Lemma.** If $X$ is an ANR closed in $E^n$, $\dim X \leq n - 1$, $x \in X$, and $V$ is any neighborhood of $x$, then there exists a neighborhood $U$ of $x$ such that $U \subseteq V$, and $U - X$ has finitely many components.

**Proof.** Let $W$ be any compact neighborhood of $x$ which is contained in $V$. By the last-stated lemma in [4], $W$ meets only finitely
many components of \(V - X\). Let \(K\) be the union of these components, and put \(U = \text{Int} \ \bar{K}\). \(U\) clearly has the required properties.

A residual set satisfying the hypotheses of the above theorem need not itself be strongly residual, even if it has dimension \(n - 1\), as the following example shows:

**Example.** Consider \(S^3\) as the one-point compactification of \(E^3\), and define subsets of \(S^3\) via a rectangular coordinate system in \(E^3\), as follows:

\[
A = \{(x, y, z) \mid z = 0, -1 \leq x \leq 0, -1 \leq y \leq 1\}, \\
B = \{(x, y, z) \mid 0 < x \leq 1, y = \sin(1/x), 0 \leq z \leq 1\}, \\
C = \{(x, y, z) \mid z = 0, 0 \leq x \leq \epsilon, -1 \leq y \leq 1\},
\]

where \(\epsilon\) is any small positive number.

\(A \cup B\) is not strongly residual in \(S^3\) since, failing to be locally connected, it cannot be an ANR (see [3]). However, \(A \cup B\) can easily be shown to be cellular, and this implies that it is residual in \(S^3\). Including the additional set \(C\) cleans matters up quite a bit; albeit tedious, it is by no means a herculean task to exhibit a pseudo-isotopy of the cube \(\{(x, y, z) \mid -1 \leq x, y, z \leq 1\}\) onto \(A \cup B \cup C\).

The question as to whether a residual ANR can fail to be strongly residual remains open.

**References**


