SOME SEMIGROUPS ON A MANIFOLD WITH BOUNDARY

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Abstract. In this paper, $S$ is an abelian semigroup on an $n$-dimensional simply connected manifold with boundary whose interior is a dense, simply connected, connected Lie group. We also assume there is a vector semigroup $V_k$ in $S$ such that the interior of $S$ misses the boundary of $V_k$, and such that $(S-GL_k)/V_k$ is a group. It is shown that if $k = n$, then $S$ is isomorphic to $V_k^n$, and if $k = 1, 2,$ or $n-1$, then $S$ is isomorphic to $V_{n-k} \times V_k$.

Introduction. In this paper we employ the language of topological semigroups, and that of transformation groups. The former may be found in [5], and the latter in [7]. Semigroup is to mean topological semigroup. If $S$ is a semigroup with identity, 1, and $N$ is a group in $S$ with 1 in $N$, then $N$ acts as a transformation semigroup on $S$ by left multiplication, and any two distinct orbits of this action are disjoint. Thus, if $M$ is a subset of $S$ which is invariant under this action, we may form the orbit space $M/N$. Whenever we say that $M/N$ is a group, we mean the operation $(Nm)(Nm') = N(mm')$ is well-defined, and makes $M/N$ into a group (algebraically speaking).

We denote the multiplicative group of positive reals by $P$, and use $P^r$ to designate the multiplicative semigroup of nonnegative reals. Referring to [2], for each positive integer $k$, we set

$$V_k = P \times P \times \cdots \times P \ (k\text{-copies}), \quad V_k^- = P^r \times P^r \times \cdots \times P^r \ (k\text{-copies})$$

and $L_k = V_k^- - V_k$.

We use $e$ to denote the zero of $V_k^r$, and obtain information about $V_k^r$, $V_k^r$, and $L_k$ from [2]; for example, $L_k$ is a connected ideal in $V_k^r$.

In what follows, $S$ is to be an abelian semigroup on an $n$-dimensional simply connected manifold with boundary such that the interior of $S$ is a dense, connected, simply connected group, $G$. We do not assume $S$ is compact. Since $G$ is dense in $S$, the identity, 1, of $G$ is the identity for $S$. We further assume that there is a $k < n+1$ such that $V_k^r \subseteq S, \ 1 \in V_k^r, \ G \cap L_k = \emptyset$, and $(S-GL_k)/V_k$ is a group. It will be shown that $S-GL_k = G$. 

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Since \( G \) is the interior of \( S \), \( H(1) \) is a Lie group \([9]\). Since \( G \) is dense and open in \( S \), it is seen that \( G = H(1) \). Further, since \( S \) is abelian, \( \text{Bd}(S) = S - G \) is an ideal in \( S \). \( G \) is a connected, simply connected, \( n \)-dimensional Lie group, so \( G \) is isomorphic to the \( n \)-dimensional vector group \([4]\). If \( A \) is a subset of \( S \), \( A^* \) denotes the closure of \( A \).

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**Preliminary results.** Suppose \( V \) is a vector group, and \( H \) is a subgroup of \( V \) such that \( V/H \) is also a vector group. If \( \pi : V \to V/H \) is the natural map, \( \pi \) is linear \([4]\), so \( H \) is a vector subspace of \( V \). This establishes

**Lemma 1.** If \( V \) is a vector group, and \( H \) is a subgroup such that \( V/H \) is also a vector group, then \( H \) is a vector subspace of \( V \).

Our first interesting result is

**Theorem 1.** \( G = S - GL_k \), and \( \text{Bd}(S) = GL_k \). Furthermore, \( \text{Bd}(S) \) is connected and is an \((n - 1)\)-manifold.

**Proof.** Since \( G \cap L_k = \emptyset \), and since \( \text{Bd}(S) \) is an ideal, \( G = S - \text{Bd}(S) \) \( \subseteq S - GL_k \). Since \( V_k \subset H(1) = G \), \( V_k \subseteq S - GL_k \). If \( t \in S - GL_k \), and if \( v \in V_k \), then \( vt \in S - GL_k \). For, if \( vt \in GL_k \), \( t \in (v^{-1}G)L_k = GL_k \). We then see that for every \( t \in S - GL_k \), \( V_k t \subset S - GL_k \). Since \( (S - GL_k)/V_k \) is a group and \( 1 \in V_k \subseteq G \subseteq S - GL_k \), it is seen from \([3]\) that \( S - GL_k \subset H(1) = G \). Therefore, \( G = S - GL_k \).

\( G = S - GL_k \), so, \( \text{Bd}(S) = S - G = GL_k \). As mentioned above, \( L_k \) is connected, so, since \( G \) is connected, \( \text{Bd}(S) = GL_k \) is connected. It then follows \([10]\) that \( \text{Bd}(S) \) is an \((n - 1)\)-manifold.

We now present

**Theorem 2.** \( G_e = S_e \), and \( G_e \) is a vector group of dimension not greater than \( n - k \). Furthermore, \( G_e \), the isotropy subgroup of \( G \) at \( e \) under left multiplication, is connected.

**Proof.** Since \( G \) is isomorphic to the \( n \)-dimensional vector group, and since \( V_k \) is a vector subgroup of \( G \), there is a vector group \( V_{n-k} \) in \( G \) such that \( V_{n-k} \times V_k \) is isomorphic to \( G \) under \((v, t) \to vt\). Thus, \( G = V_{n-k}V_k \), so \( GL_k = (V_{n-k}V_k)L_k = V_{n-k}(V_kL_k) \). But, \( L_k \) is an ideal in \( V_k \), so, since \( 1 \in V_k \), \( V_kL_k \subset L_k \subset V_kL_k \). Thus, \( V_kL_k = L_k \), and \( GL_k = V_{n-k}L_k \). From this we see that \((v, t) \to vt\) maps \( V_{n-k} \times V_k \) homomorphically onto \( G \cup GL_k = S \).

Since \( e \) is the zero of \( V_k \), since \( G = V_{n-k}V_k \), and since \( S = V_{n-k}V_k \), we readily see that \( G_e = V_{n-k}e = S_e \). This is the first part of the theorem.
Now, \( e \) is an idempotent in \( S \), and there is a one-parameter semigroup in \( V_k \subset S \) which has \( e \) as its zero. Thus, \( Se = Ge \) is a deformation retract of \( S \). Hence, \( Ge \) is closed in \( S \), so it is locally compact. Also, since \( S \) is abelian, \( Ge \) is algebraically a group with identity \( e = 1e \). Therefore [1], \( Ge \) is a topological group. Furthermore, \( v \mapsto ve \) is a homomorphism from the locally compact, Lindelöf, Hausdorff topological group \( V_{n-k} \) onto \( Ge \). Hence, the map is open, and \( Ge \) is isomorphic to \((V_{n-k})/(V_{n-k})_e\), where \((V_{n-k})_e\) is the isotropy subgroup of \( V_{n-k} \) at \( e \). Since \((V_{n-k})_e\) is a closed subgroup of \( V_{n-k} \), \((V_{n-k})/(V_{n-k})_e\) is isomorphic to a product of a vector group \( V_p \) and a toral group \( T_q \), with \( p + q < n - k + 1 \) [4]. Then, \( Ge \cong (V_{n-k})/(V_{n-k})_e \cong V_p \times T_q \).

Letting \( \pi_1 \) denote the fundamental group functor, we have
\[
\pi_1(Se) = \pi_1(V_p \times T_q) = \pi_1(V_p) \oplus \pi_1(T_q)
\]
\[
= \pi_1(T_q) = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \quad (q\text{-copies}).
\]

However, \( Ge = Se \) is a deformation retract of the simply connected space \( S \), so \( \pi_1(Se) = 0 \). Therefore, \( q = 0 \), and \( Ge \) is isomorphic to \( V_p \), \( p < n - k + 1 \). Hence, \( Ge = Se \) is a vector group of dimension not greater than \( n - k \).

\( G \) is a vector group, so \( G \) is a locally compact, Lindelöf, Hausdorff topological group. Further, \( g \mapsto ge \) is a homomorphism of \( G \) onto \( Ge \), so, as above, \( Ge \) is isomorphic to \( G/Ge \). Hence, \( Ge \) is a subgroup of the vector group \( G \) such that \( G/Ge \) is a vector group. Hence, by Lemma 1, \( Ge \) is a vector group, and therefore connected, which concludes the proof of the theorem.

We remark that the proof above can also be used to show that \((V_{n-k})_e\) is connected.

We now prove a theorem which leads to our characterization theorems. It is

**Theorem 3.** The following are equivalent:
\( (i) \) \( S \) is isomorphic to \( V_{n-k} \times V_k^r \) under the map \((v, t) \mapsto vt\).
\( (ii) \) \( v \mapsto ve \) is an isomorphism from \( V_{n-k} \) onto \( Ge \).
\( (iii) \) \( \dim Ge = n - k \).

**Proof.** From the proof of Theorem 2, \((v, t) \mapsto vt\) is a homomorphism from \( V_{n-k} \times V_k^r \) onto \( S \), and an isomorphism from \( V_{n-k} \times V_k \) onto \( G \). Also, \( v \mapsto ve \) is an open homomorphism from \( V_{n-k} \) onto \( Ge \).

(i\( \implies \)) If \((v, t) \mapsto vt\) is an isomorphism, it follows that \( v \mapsto ve \) must be 1-1 from \( V_{n-k} \) onto \( Ge \). Thus, \( v \mapsto ve \) is an isomorphism from \( V_{n-k} \) onto \( Ge \).
(ii$\Rightarrow$iii) If $v\rightarrow ve$ is an iseomorphism, then $V_{n-k}$ is homeomorphic to $Ge$, so $\dim Ge = \dim V_{n-k} = n - k$.

(iii$\Rightarrow$ii) It is shown in the proof of Theorem 2 that $v\rightarrow ve$ is an open homomorphism from $V_{n-k}$ onto $Ge$, and that $Ge$ is iseomorphic to $(V_{n-k})/(V_{n-k})_e$. Thus, $n - k = \dim Ge = \dim [(V_{n-k})/(V_{n-k})_e]$. But, \[8\], $\dim [(V_{n-k})/(V_{n-k})_e] = \dim V_{n-k} - \dim (V_{n-k})_e = (n - k) - \dim (V_{n-k})_e$.

Then, $n - k = (n - k) - \dim (V_{n-k})_e$, so $\dim (V_{n-k})_e = 0$. Hence, $(V_{n-k})_e = \{1\}$, so $v\rightarrow ve$ is 1-1, and is thus an iseomorphism from $V_{n-k}$ onto $Ge$.

(ii$\Rightarrow$i) We already know that $(v, t)\rightarrow vu$ is a homomorphism from $V_{n-k} \times V_k$ onto $S$. We now show that it is 1-1.

Suppose $v, v' \in V_{n-k}$ and $t, t' \in V_k$ such that $vt = v't'$. Then, since $e$ is the zero for $V_k$, $ve = (vt)e = (v't')e = ve$. Thus, $v = v'$, because $s\rightarrow se$ is an iseomorphism from $V_{n-k}$ onto $Ge$. Then, $t = v^{-1}(vt) = v^{-1}(v't') = v^{-1}v = t$. Hence, $(v, t) = (v', t')$, and our map is 1-1.

To show that our map is an iseomorphism, it is now sufficient to show that if $\{v_p\}$ and $\{t_p\}$ are nets in $V_{n-k}$ and $V_k$ respectively such that $v_p t_p \rightarrow vt$ for some $v \in V_{n-k}$ and $t \in V_k$, then $(v_p, t_p) \rightarrow (v, t)$. But, if $v_p t_p \rightarrow vt$, then $v_p e = (v_p t_p)e \rightarrow (vt)e = ve$. Since $s\rightarrow se$ is an iseomorphism, $v_p \rightarrow w$. $V_{n-k}$ is a topological group, so $v_p^{-1} \rightarrow w^{-1}$, and $t_p = v_p^{-1}(v_p t_p) \rightarrow v^{-1}(vt) = t$. Hence, $(v_p, t_p) \rightarrow (v, t)$, and we have completed the proof of the theorem.

**Principal results.** We now give our characterizations of $S$. The first of these is

**Theorem 4.** If $k = n$, $S$ is iseomorphic to $V_k^-$. 

**Proof.** If $k = n$, $V_k$ is an $n$-dimensional vector subgroup of $G$. Hence, $V_k = G$, so $S = G* = (V_k)^* = V_k^- [2]$, and we see that $S = V_k^- = V_k^-$. 

**Theorem 5.** If $k = n - 1$, and $e$ is not a zero for $S$, then $S$ is iseomorphic to $V_1 \times V_{n-1}^-$. 

**Proof.** $Ge = V_1 e$, so $\dim Ge \leq \dim V_1 = 1$. If $\dim Ge = 0$, then, since $Ge$ is connected and nonempty, $Ge = \{e\}$. But, by Theorem 2, $Se = Ge = \{e\}$, so $e$ is a zero for $S$. This contradiction implies that $\dim Ge = 1$, which, by Theorem 3, yields the result.

**Theorem 6.** If $k = 1$, $S$ is iseomorphic to $V_{n-1} \times V_1^-$. 

**Proof.** $S$ is an $n$-manifold with boundary, and, by Theorem 1, $\text{Bd}(S)$ is an $(n - 1)$-manifold, and $\text{Bd}(S) = GL_1$. Since $L_1 = \{e\}$, $GL_1 = Ge$. Thus, $\dim Ge = n - 1$, and the result follows from Theorem 3.

The last characterization we have obtained to date is
Theorem 7. If $k = 2$, $S$ is isomorphic to $V_{n-2} \times V_2$.

Proof. From Theorem 1, $\text{Bd}(S) = GL_2$, and from the proof of Theorem 2, $G = V_{n-2} V_2$, and $S = V_{n-2} V_2$. From [2], we may find idempotents $e_1$ and $e_2$ in $L_2$ such that $e_1 \neq e_2$, $e_1 \neq e \neq e_2$, $e = e_1 e_2$, and $L_2 = V_2 e_1 \cup V_2 e_2$. We then see that

$$\text{Bd}(S) = GL_2 = G(V_{n-2} e_1 \cup V_{n-2} e_2) \subset GV_{n-2} e_1 \cup GV_{n-2} e_2$$

$$\subset GL_2 \cup GL_2 = GL_2 = \text{Bd}(S).$$

Therefore, $\text{Bd}(S) = GV_{n-2} e_1 \cup GV_{n-2} e_2$. We have, for $i = 1, 2$, $GV_{n-2} e_i = (V_{n-2} V_2) e_i = (V_{n-2} V_2) e_i = S e_i$. Thus, $\text{Bd}(S) = S e_1 \cup S e_2$. Since each of $e_1$ and $e_2$ is an idempotent in $S$, each of $S e_1$ and $S e_2$ is a retract of $S$ and hence closed in $S$, and thus in $\text{Bd}(S)$.

For $i = 1, 2$, we see that $S e_i \subset (G e_i)^*$, because $G^* = S$. But, $G \subset S$, so $S e_i \subset S e_i$. Since $S e_i$ is closed, it follows that $(G e_i)^* = S e_i$. Now, $S e_i = (V_{n-2} V_2) e_i = V_{n-2} (V_2 e_i)$. However, [2] gives $V_2 e_i = V_{n-2} e_i \cup \{e\}$, so we have $S e_i = V_{n-2} (V_2 e_i \cup \{e\}) \subset V_{n-2} V_2 e_i \cup V_{n-2} e_i \subset G e_i$. Therefore, $S e_i = G e_i \cup G e$.

If $i \neq j$, $G e_i \cap S e_j = \emptyset$. For, if $g \in G$ such that $g e_i \in S e_j$, $e_i \in S e_j$, and there is an $s \in S$ such that $e_i = s e_j$. Then, $e_i = e_i^2 = (s e_j) e_i = s e = e$, which is a contradiction. Therefore, $G e_i \cap S e_j = \emptyset$. Since $G e \subset S e_j$, it follows that for each $i = 1, 2$, $G e_i \cap G e = \emptyset$. Hence, $S e_i = G e_i$. Ge.

From above, $\text{Bd}(S) = S e_1 \cup S e_2 = G e_1 \cup G e \cup G e_2$, and no two of these intersect. Thus, $\text{Bd}(S) - G e_1 = G e \cup G e_2 = S e_2$. $S e_2$ is closed in $\text{Bd}(S)$, so $G e_1$ is open in $\text{Bd}(S)$. Letting $\partial(G e_1)$ be the boundary of $G e_1$ in $\text{Bd}(S)$, we see, since $G e_1$ is open in $\text{Bd}(S)$, that $\partial(G e_1) = (G e_1)^* - G e_1 = S e_1 - G e_1 = G e$. Furthermore, $e_2 \in (G e_1)^* = S e_1$ from above, so we see that $G e_1$ is a nonempty, nondense, open subset of the $(n - 1)$-manifold $\text{Bd}(S)$. Hence [6], $\dim G e = \dim \partial(G e_1) = (n - 1) - 1 = n - 2$. The theorem follows from Theorem 3.

Conclusion. The results of this paper give a partial characterization of $S$. It should be observed that in all cases presented, $S$ cannot be compact. One wonders whether or not there is a compact $S$ which satisfies our criterion. One also queries whether or not $S$ is isomorphic to $V_{n-k} \times V_k$ for all $k$. We are presently working on these problems.

Bibliography


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