SOME SEMIGROUPS ON A MANIFOLD WITH BOUNDARY[1]

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ABSTRACT. In this paper, S is an abelian semigroup on an n-dimensional simply connected manifold with boundary whose interior is a dense, simply connected, connected Lie group. We also assume there is a vector semigroup V_k^- in S such that the interior of S misses the boundary of V_k^-, and such that $(S-GL_k)/V_k$ is a group. It is shown that if $k=n$, then S is iseomorphic to V_n^-, and if $k=1,2$, or $n-1$, then S is iseomorphic to $V_{n-k}\times V_k^-$.

Introduction. In this paper we employ the language of topological semigroups, and that of transformation groups. The former may be found in [5], and the latter in [7]. Semigroup is to mean topological semigroup. If S is a semigroup with identity, 1, and N is a group in S with 1 in N, then N acts as a transformation semigroup on S by left multiplication, and any two distinct orbits of this action are disjoint. Thus, if M is a subset of S which is invariant under this action, we may form the orbit space M/N. Whenever we say that M/N is a group, we mean the operation $(Nm)(Nm')=N(mm')$ is well-defined, and makes M/N into a group (algebraically speaking).

We denote the multiplicative group of positive reals by P, and use P^- to designate the multiplicative semigroup of nonnegative reals. Referring to [2], for each positive integer k, we set

$$V_k=P\times P\times \cdots \times P \ (k\text{-copies}),\quad V_k^-=P^-\times P^-\times \cdots \times P^- \ (k\text{-copies})$$

and $L_k=V_k^- - V_k$.

We use e to denote the zero of V_k^-, and obtain information about V_k^-, V_k, and L_k from [2]; for example, L_k is a connected ideal in V_k^-.

In what follows, S is to be an abelian semigroup on an n-dimensional simply connected manifold with boundary such that the interior of S is a dense, connected, simply connected group, G. We do not assume S is compact. Since G is dense in S, the identity, 1, of G is the identity for S. We further assume that there is a $k<n+1$ such that $V_k^-\subset S$, $1\in V_k^-$, $G\cap L_k=\varnothing$, and $(S-GL_k)/V_k$ is a group. It will be shown that $S-GL_k=G$.

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Since G is the interior of S, $H(1)$ is a Lie group [9]. Since G is dense and open in S, it is seen that $G=H(1)$. Further, since S is abelian, $\mathrm{Bd}(S)=S-G$ is an ideal in S. G is a connected, simply connected, n-dimensional Lie group, so G is iseomorphic to the n-dimensional vector group [4]. If A is a subset of S, A^* denotes the closure of A.

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Preliminary results. Suppose V is a vector group, and H is a subgroup of V such that V/H is also a vector group. If $\pi: V\rightarrow V/H$ is the natural map, π is linear [4], so H is a vector subspace of V. This establishes

LEMMA 1. *If V is a vector group, and H is a subgroup such that V/H is also a vector group, then H is a vector subspace of V.*

Our first interesting result is

THEOREM 1. *$G=S-GL_k$, and $\mathrm{Bd}(S)=GL_k$. Furthermore, $\mathrm{Bd}(S)$ is connected and is an $(n-1)$-manifold.*

PROOF. Since $G\cap L_k=\emptyset$, and since $\mathrm{Bd}(S)$ is an ideal, $G=S-\mathrm{Bd}(S)\subset S-GL_k$. Since $V_k\subset H(1)=G$, $V_k\subset S-GL_k$. If $t\in S-GL_k$, and if $v\in V_k$, then $vt\in S-GL_k$. For, if $vt\in GL_k$, $t\in(v^{-1}G)L_k=GL_k$. We then see that for every $t\in S-GL_k$, $V_kt\subset S-GL_k$. Since $(S-GL_k)/V_k$ is a group and $1\in V_k\subset G\subset S-GL_k$, it is seen from [3] that $S-GL_k\subset H(1)=G$. Therefore, $G=S-GL_k$.

$G=S-GL_k$, so, $\mathrm{Bd}(S)=S-G=GL_k$. As mentioned above, L_k is connected, so, since G is connected, $\mathrm{Bd}(S)=GL_k$ is connected. It then follows [10] that $\mathrm{Bd}(S)$ is an $(n-1)$-manifold.

We now present

THEOREM 2. *$Ge=Se$, and Ge is a vector group of dimension not greater than $n-k$. Furthermore, G_e, the isotropy subgroup of G at e under left multiplication, is connected.*

PROOF. Since G is iseomorphic to the n-dimensional vector group, and since V_k is a vector subgroup of G, there is a vector group V_{n-k} in G such that $V_{n-k}\times V_k$ is iseomorphic to G under $(v, t)\rightarrow vt$. Thus, $G=V_{n-k}V_k$, so $GL_k=(V_{n-k}V_k)L_k=V_{n-k}(V_kL_k)$. But, L_k is an ideal in V_k^-, so, since $1\in V_k$, $V_kL_k\subset L_k\subset V_kL_k$. Thus, $V_kL_k=L_k$, and $GL_k=V_{n-k}L_k$. From this we see that $(v, t)\rightarrow vt$ maps $V_{n-k}\times V_k^-$ homomorphically onto $G\cup GL_k=S$.

Since e is the zero of V_k^-, since $G=V_{n-k}V_k$, and since $S=V_{n-k}V_k^-$, we readily see that $Ge=V_{n-k}e=Se$. This is the first part of the theorem.

Now, e is an idempotent in S, and there is a one-parameter semigroup in $V_k^- \subset S$ which has e as its zero. Thus, $Se = Ge$ is a deformation retract of S. Hence, Ge is closed in S, so it is locally compact. Also, since S is abelian, Ge is algebraically a group with identity $e = 1e$. Therefore [1], Ge is a topological group. Furthermore, $v \rightarrow ve$ is a homomorphism from the locally compact, Lindelöf, Hausdorff topological group V_{n-k} onto Ge. Hence, the map is open, and Ge is iseomorphic to $(V_{n-k})/(V_{n-k})_e$, where $(V_{n-k})_e$ is the isotropy subgroup of V_{n-k} at e. Since $(V_{n-k})_e$ is a closed subgroup of V_{n-k}, $(V_{n-k})/(V_{n-k})_e$ is iseomorphic to a product of a vector group V_p and a toral group T_q, with $p+q<n-k+1$ [4]. Then, $Ge \cong (V_{n-k})/(V_{n-k})_e \cong V_p \times T_q$.

Letting π_1 denote the fundamental group functor, we have

$$\begin{aligned}\pi_1(Ge) &= \pi_1(V_p \times T_q) = \pi_1(V_p) \oplus \pi_1(T_q) \\ &= \pi_1(T_q) = Z \oplus Z \oplus \cdots \oplus Z \quad (q\text{-copies}).\end{aligned}$$

However, $Ge = Se$ is a deformation retract of the simply connected space S, so $\pi_1(Ge) = 0$. Therefore, $q = 0$, and Ge is iseomorphic to V_p, $p < n-k+1$. Hence, $Ge = Se$ is a vector group of dimension not greater than $n-k$.

G is a vector group, so G is a locally compact, Lindelöf, Hausdorff topological group. Further, $g \rightarrow ge$ is a homomorphism of G onto Ge, so, as above, Ge is iseomorphic to G/G_e. Hence, G_e is a subgroup of the vector group G such that G/G_e is a vector group. Hence, by Lemma 1, G_e is a vector group, and therefore connected, which concludes the proof of the theorem.

We remark that the proof above can also be used to show that $(V_{n-k})_e$ is connected.

We now prove a theorem which leads to our characterization theorems. It is

THEOREM 3. *The following are equivalent*:

(i) *S is iseomorphic to $V_{n-k} \times V_k^-$ under the map $(v, t) \rightarrow vt$.*

(ii) *$v \rightarrow ve$ is an iseomorphism from V_{n-k} onto Ge.*

(iii) $\dim Ge = n-k$.

PROOF. From the proof of Theorem 2, $(v, t) \rightarrow vt$ is a homomorphism from $V_{n-k} \times V_k^-$ onto S, and an iseomorphism from $V_{n-k} \times V_k$ onto G. Also, $v \rightarrow ve$ is an open homomorphism from V_{n-k} onto Ge.

(i$\Rightarrow$ii) If $(v, t) \rightarrow vt$ is an iseomorphism, it follows that $v \rightarrow ve$ must be 1-1 from V_{n-k} onto Ge. Thus, $v \rightarrow ve$ is an iseomorphism from V_{n-k} onto Ge.

(ii⇒iii) If $v \to ve$ is an iseomorphism, then V_{n-k} is homeomorphic to Ge, so $\dim Ge = \dim V_{n-k} = n-k$.

(iii⇒ii) It is shown in the proof of Theorem 2 that $v \to ve$ is an open homomorphism from V_{n-k} onto Ge, and that Ge is iseomorphic to $(V_{n-k})/(V_{n-k})_e$. Thus, $n-k = \dim Ge = \dim[(V_{n-k})/(V_{n-k})_e]$. But, [8], $\dim[(V_{n-k})/(V_{n-k})_e] = \dim V_{n-k} - \dim(V_{n-k})_e = (n-k) - \dim(V_{n-k})_e$. Then, $n-k = (n-k) - \dim(V_{n-k})_e$, so $\dim(V_{n-k})_e = 0$. Hence, $(V_{n-k})_e = \{1\}$, so $v \to ve$ is 1-1, and is thus an iseomorphism from V_{n-k} onto Ge.

(ii⇒i) We already know that $(v, t) \to vt$ is a homomorphism from $V_{n-k} \times V_k^-$ onto S. We now show that it is 1-1.

Suppose $v, v' \in V_{n-k}$ and $t, t' \in V_k^-$ such that $vt = v't'$. Then, since e is the zero for V_k^-, $ve = (vt)e = (v't')e = v'e$. Thus, $v = v'$, because $s \to se$ is an iseomorphism from V_{n-k} onto Ge. Then, $t = v^{-1}(vt) = v^{-1}(v't') = v^{-1}(vt') = t'$. Hence, $(v, t) = (v', t')$, and our map is 1-1.

To show that our map is an iseomorphism, it is now sufficient to show that if $\{v_\rho\}$ and $\{t_\rho\}$ are nets in V_{n-k} and V_k^- respectively such that $v_\rho t_\rho \to vt$ for some $v \in V_{n-k}$ and $t \in V_k^-$, then $(v_\rho, t_\rho) \to (v, t)$. But, if $v_\rho t_\rho \to vt$, then $v_\rho e = (v_\rho t_\rho)e \to (vt)e = ve$. Since $s \to se$ is an iseomorphism, $v_\rho \to v$. V_{n-k} is a topological group, so $v_\rho^{-1} \to v^{-1}$, and $t_\rho = v_\rho^{-1}(v_\rho t_\rho) \to v^{-1}(vt) = t$. Hence, $(v_\rho, t_\rho) \to (v, t)$, and we have completed the proof of the theorem.

Principal results. We now give our characterizations of S. The first of these is

THEOREM 4. *If $k = n$, S is iseomorphic to V_n^-.*

PROOF. If $k = n$, V_k is an n-dimensional vector subgroup of G. Hence, $V_k = G$, so $S = G^* = (V_k)^* = V_k^-$ [2], and we see that $S = V_k^- = V_n^-$.

THEOREM 5. *If $k = n-1$, and e is not a zero for S, then S is iseomorphic to $V_1 \times V_{n-1}^-$.*

PROOF. $Ge = V_1 e$, so $\dim Ge \leqq \dim V_1 = 1$. If $\dim Ge = 0$, then, since Ge is connected and nonempty, $Ge = \{e\}$. But, by Theorem 2, $Se = Ge = \{e\}$, so e is a zero for S. This contradiction implies that $\dim Ge = 1$, which, by Theorem 3, yields the result.

THEOREM 6. *If $k = 1$, S is iseomorphic to $V_{n-1} \times V_1^-$.*

PROOF. S is an n-manifold with boundary, and, by Theorem 1, $\mathrm{Bd}(S)$ is an $(n-1)$-manifold, and $\mathrm{Bd}(S) = GL_1$. Since $L_1 = \{e\}$, $GL_1 = Ge$. Thus, $\dim Ge = n-1$, and the result follows from Theorem 3.

The last characterization we have obtained to date is

THEOREM 7. *If $k=2$, S is iseomorphic to $V_{n-2}\times V_2^-$.*

PROOF. From Theorem 1, $\mathrm{Bd}(S)=GL_2$, and from the proof of Theorem 2, $G=V_{n-2}V_2$, and $S=V_{n-2}V_2^-$. From [2], we may find idempotents e_1 and e_2 in L_2 such that $e_1\neq e_2$, $e_1\neq e\neq e_2$, $e=e_1e_2$, and $L_2=V_2^-e_1\cup V_2^-e_2$. We then see that

$$\begin{aligned}\mathrm{Bd}(S) &= GL_2 = G(V_2^-e_1\cup V_2^-e_2)\subset GV_2^-e_1\cup GV_2^-e_2\\ &\subset GL_2\cup GL_2 = GL_2 = \mathrm{Bd}(S).\end{aligned}$$

Therefore, $\mathrm{Bd}(S)=GV_2^-e_1\cup GV_2^-e_2$. We have, for $i=1, 2$, $GV_2^-e_i=(V_{n-2}V_2)V_2^-e_i=(V_{n-2}V_2^-)e_i=Se_i$. Thus, $\mathrm{Bd}(S)=Se_1\cup Se_2$. Since each of e_1 and e_2 is an idempotent in S, each of Se_1 and Se_2 is a retract of S and hence closed in S, and thus in $\mathrm{Bd}(S)$.

For $i=1, 2$, we see that $Se_i\subset(Ge_i)^*$, because $G^*=S$. But, $G\subset S$, so $Ge_i\subset Se_i$. Since Se_i is closed, it follows that $(Ge_i)^*=Se_i$. Now, $Se_i=(V_{n-2}V_2^-)e_i=V_{n-2}(V_2^-e_i)$. However, [2] gives $V_2^-e_i=V_2e_i\cup\{e\}$, so we have $Se_i=V_{n-2}[V_2e_i\cup\{e\}]\subset V_{n-2}V_2e_i\cup V_{n-2}e\subset Ge_i\cup Ge\subset Se_i$. Therefore, $Se_i=Ge_i\cup Ge$.

If $i\neq j$, $Ge_i\cap Se_j=\varnothing$. For, if $g\in G$ such that $ge_i\in Se_j$, $e_i\in Se_j$, and there is an $s\in S$ such that $e_i=se_j$. Then, $e_i=e_i^2=(se_j)e_i=se$, so, $e=e_ie_j=(se)e_j=se=e_i$, which is a contradiction. Therefore, $Ge_i\cap Se_j=\varnothing$. Since $Ge\subset Se_j$, it follows that for each $i=1, 2$ $Ge_i\cap Ge=\varnothing$. Hence, $Se_i-Ge_i=Ge$.

From above, $\mathrm{Bd}(S)=Se_1\cup Se_2=Ge_1\cup Ge\cup Ge_2$, and no two of these intersect. Thus, $\mathrm{Bd}(S)-Ge_1=Ge\cup Ge_2=Se_2$. Se_2 is closed in $\mathrm{Bd}(S)$, so Ge_1 is open in $\mathrm{Bd}(S)$. Letting $\partial(Ge_1)$ be the boundary of Ge_1 in $\mathrm{Bd}(S)$, we see, since Ge_1 is open in $\mathrm{Bd}(S)$, that $\partial(Ge_1)=(Ge_1)^*-Ge_1=Se_1-Ge_1=Ge$. Furthermore, $e_2\notin(Ge_1)^*=Se_1$ from above, so we see that Ge_1 is a nonempty, nondense, open subset of the $(n-1)$-manifold $\mathrm{Bd}(S)$. Hence [6], $\dim Ge=\dim\partial(Ge_1)=(n-1)-1=n-2$. The theorem follows from Theorem 3.

CONCLUSION. The results of this paper give a partial characterization of S. It should be observed that in all cases presented, S cannot be compact. One wonders whether or not there is a compact S which satisfies our criterion. One also queries whether or not S is iseomorphic to $V_{n-k}\times V_k^-$ for all k. We are presently working on these problems.

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