SHORTER NOTES

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A NOTE ON THE FAILURE OF THE RELATIVIZED ENUMERATION THEOREM IN RECURSIVE FUNCTION THEORY

RICHARD A. PLATEK

For any set A of nonnegative integers let ψ_A, the semicharacteristic function of A, be the partial function that is 0 on A and undefined off A.

THEOREM. *Let A be any many-one complete Σ_1^1 set. Then ψ_A has the following properties*:

(i) *The total functions recursive in ψ_A are exactly the hyperarithmetic functions.*

(ii) *There is no total function in ψ_A's Turing degree.*

(iii) *There is no 2-ary function partial recursive in ψ_A which enumerates all the 1-ary functions partial recursive in ψ_A.*

PROOF. Let ψ_A be as above. A function Φ will be partial recursive in ψ_A if there is an r.e. relation R with

$$\Phi(x_1, \cdots, x_n) \cong y \Leftrightarrow \exists u[R(x_1, \cdots, x_n, y, u) \wedge [u] \subseteq A] \tag{1}$$

where $[u]$ is the uth finite set in some recursive coding of finite sets. Since A is Σ_1^1 (1) shows that any Φ partial recursive in ψ_A has Σ_1^1 graph; furthermore since A is many-one complete for Σ_1^1 sets we have that all Φ with Σ_1^1 graphs are partial recursive in ψ_A. Since a total function has Σ_1^1 graph if it is hyperarithmetic this proves (i). (ii) follows from (i) since there is no highest degree among hyperarithmetic functions. To show (iii) we first show

LEMMA. *Any partial function with Σ_1^1 graph can be extended by a hyperarithmetic function.*

PROOF. Let Φ be a partial function such that its graph S_0 is Σ_1^1 where

$$\Phi(x) \cong y \Leftrightarrow S_0(x, y).$$

Let S_1 be

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$$S_1(x, y) \Leftrightarrow \forall z[S_0(x, z) \Rightarrow y = z].$$

S_1 is Π_1^1 and $\forall x \exists y S_1(x, y)$. By the single-valueness theorem for Π_1^1 predicates there is a hyperarithmetic function α with $\forall x S_1(x, \alpha(x))$ and α extends Φ.■

To continue the proof we note that if there were a 2-ary enumerating function recursive in ψ_A it would by the lemma have a hyperarithmetic extension which would enumerate all the hyperarithmetic functions and that is a contradiction.

We note that if Σ_1^1 is replaced by Π_1^1 in the statement of the theorem then (i) and (ii) are proved as above but (iii) is false. The degree of the resulting function is incomparable with the degree of the original function.

CORNELL UNIVERSITY, ITHACA, NEW YORK 14850