SHORTER NOTES

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A NOTE ON THE FAILURE OF THE RELATIVIZED ENUMERATION THEOREM IN RECURSIVE FUNCTION THEORY

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For any set $A$ of nonnegative integers let $\psi_A$, the semicharacteristic function of $A$, be the partial function that is 0 on $A$ and undefined off $A$.

**Theorem.** Let $A$ be any many-one complete $\Sigma_1^1$ set. Then $\psi_A$ has the following properties:

(i) The total functions recursive in $\psi_A$ are exactly the hyperarithmetic functions.

(ii) There is no total function in $\psi_A$'s Turing degree.

(iii) There is no 2-ary function partial recursive in $\psi_A$ which enumerates all the 1-ary functions partial recursive in $\psi_A$.

**Proof.** Let $\psi_A$ be as above. A function $\Phi$ will be partial recursive in $\psi_A$ if there is an r.e. relation $R$ with

\begin{equation}
\Phi(x_1, \ldots, x_n) \leq y \iff \exists u[R(x_1, \ldots, x_n, y, u) \land [u] \subseteq A]
\end{equation}

where $[u]$ is the $u$th finite set in some recursive coding of finite sets. Since $A$ is $\Sigma_1^1$ (1) shows that any $\Phi$ partial recursive in $\psi_A$ has $\Sigma_1^1$ graph; furthermore since $A$ is many-one complete for $\Sigma_1^1$ sets we have that all $\Phi$ with $\Sigma_1^1$ graphs are partial recursive in $\psi_A$. Since a total function has $\Sigma_1^1$ graph if it is hyperarithmetic this proves (i). (ii) follows from (i) since there is no highest degree among hyperarithmetic functions. To show (iii) we first show

**Lemma.** Any partial function with $\Sigma_1^1$ graph can be extended by a hyperarithmetic function.

**Proof.** Let $\Phi$ be a partial function such that its graph $S_0$ is $\Sigma_1^1$ where

\[ \Phi(x) \leq y \iff S_0(x, y). \]

Let $S_1$ be

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$S_1(x, y) \leftrightarrow \forall z [S_0(x, z) \Rightarrow y = z]$.

$S_1$ is $\Pi_1^1$ and $\forall x \exists y S_1(x, y)$. By the single-valueness theorem for $\Pi_1^1$ predicates there is a hyperarithmetic function $\alpha$ with $\forall x S_1(x, \alpha(x))$ and $\alpha$ extends $\Phi$.

To continue the proof we note that if there were a 2-ary enumerating function recursive in $\psi_\alpha$ it would by the lemma have a hyperarithmetic extension which would enumerate all the hyperarithmetic functions and that is a contradiction.

We note that if $\Sigma_1^1$ is replaced by $\Pi_1^1$ in the statement of the theorem then (i) and (ii) are proved as above but (iii) is false. The degree of the resulting function is incomparable with the degree of the original function.

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