CHAINS WHICH ARE COSET SPACES OF tl-GROUPS

ROBERT L. MADELL[1]

ABSTRACT. Let G be a lattice-ordered group with a topology with which G is a Hausdorff topological group and topological lattice. Let N be a closed prime convex l-subgroup of G and let G/N denote the topological chain of right cosets of N. It is shown that if G is locally compact, locally connected, or locally convex then G/N is either discrete or has precisely the interval topology.

1. **Introduction.** A *topological lattice-ordered group* (tl-group) is a triple $(G, \leqq, T)$ (henceforth denoted simply by G) such that (i) $(G, \leqq)$ is a lattice-ordered group (l-group), (ii) (G, T) is a topological group, and (iii) $(|G|, \leqq, T)$ is a topological lattice, where $|G|$ denotes the underlying set of the group G. (The spaces throughout this note are all Hausdorff.) Thus for example, any l-group with the discrete topology and any totally ordered group with the interval topology are tl-groups. We obtain a tl-group if we partially order real Euclidean n-space as the usual topological group by $(x_1, x_2, \cdots, x_n) \geqq (y_1, y_2, \cdots, y_n)$ if and only if $x_i \geqq y_i$, $i=1, 2, \cdots, n$. If the l-group of all order preserving permutations of the totally ordered set of real numbers is given the point-open topology it also is a tl-group.

Let e denote the group identity of a tl-group G. If N_α is a (topologically) closed prime convex l-subgroup of G then the space of right cosets G/N_α (with the projection topology) is a totally ordered topological lattice (a *topological chain*) and the natural projection map $\pi_\alpha: G \to G/N_\alpha$ is a continuous, open, lattice homomorphism. If $\{N_\alpha | \alpha \in A\}$ is a collection of closed prime convex l-subgroups of G with $\bigcap_{\alpha\in A} N_\alpha = \{e\}$ and if $\prod_{\alpha \in A} (G/N_\alpha)$ is ordered componentwise and given the Cartesian topology then the natural map $\pi: G \to \prod_{\alpha\in A} (G/N_\alpha)$ is a one-to-one, continuous, lattice homomorphism. Necessary and sufficient conditions were given in [6] for $\pi: G \to G\pi$ ($G\pi$ with the inherited topology) to be a homeomorphism. It is thus of interest to find conditions on G which guarantee that each G/N_α

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has some simple topology. We show in this note that if G is locally compact, locally connected, or locally convex then each nondiscrete G/N_α has precisely the interval topology. We also show that if C is a totally ordered set order isomorphic to the chain of right cosets of some prime convex l-subgroup of some l-group, then C with the interval topology (or, of course, the discrete topology) is isomorphic as a topological chain to the chain of right cosets of some closed prime convex l-subgroup of some tl-group. The reader is referred to [2], [3], [6] for background information in l-groups and tl-groups.

2. **Main results.** A topological chain C will be said to be *nonsectarian* if for each nonendpoint $c\in C$ and each neighborhood U of c, U contains elements a, b with $a<c<b$. We also require that if c is an endpoint, $\{c\}$ is not open. We shall show below that a nonsectarian topological chain which is either locally compact, locally connected, or locally convex has precisely the interval topology. The following result will therefore be useful.

THEOREM 1. *Let G be a tl-group and let N be a closed prime convex l-subgroup of G. Then if G/N is not discrete it is nonsectarian.*

PROOF. We assume that G/N is not discrete. In particular, $G\neq N$ so G/N has no endpoints. It suffices by homogeneity [6] to consider neighborhoods U^* of the coset N. Let $U^*=\{Nu\mid u\in U\}$ where U is open in G. Suppose by way of contradiction that say $N\leqq U^*$. Then if $V^*=\{Nu^{-1}\mid u\in U\}$ it is easy to see that V^* is a neighborhood of the coset N and $V^*\cap U^*=\{N\}$. Then by homogeneity, G/N is discrete for the contradiction.

Recall that a topological lattice L is said to be *locally convex* if whenever $x\in L$ and U is a neighborhood of x there exists an open convex set V with $x\in V\subseteq U$. The following result is easy to verify once it is recalled that topological chains have at least the interval topology. (In fact, it is easy to see that an ordered set with a topology T is a topological chain if and only if T lies between the interval and discrete topologies.)

THEOREM 2. *Let C be a nonsectarian topological chain which is either locally connected or locally convex. Then C has precisely the interval topology.*

THEOREM 3. *Let C be a locally compact nonsectarian topological chain. Then C has precisely the interval topology.*

PROOF. Pick a nonendpoint $c\in C$ (a similar argument holds for endpoints) and let U be a neighborhood of c. Since C is locally compact and Hausdorff it is regular so let P be a neighborhood of c such that $c\in P\subseteq P^c\subseteq U$ with P^c (the closure of P) compact.

Since C is nonsectarian, C is *order dense* (if $a, b\in C$ with $a<b$ there exists $d\in C$ with $a<d<b$). Thus $\{P\}\cup\{l(a)\mid a<c\}\cup\{u(a)\mid a>c\}$ is an open cover for P^c where $l(a)=\{x\in C\mid x<a\}$ and $u(a)=\{x\in C\mid x>a\}$. By compactness we can find $a, b\in C$, $a<c<b$, with $P^c\subseteq P\cup l(a)\cup u(b)$.

Define $P^*=\{x\in C\mid a<x<b\}\cap P\subseteq U$ so P^* is a neighborhood of c. It will suffice to show that $(P^*)^c$ is convex. Suppose by way of contradiction that $p_1, p_2\in(P^*)^c$ and $x\notin(P^*)^c$ although $p_1<x<p_2$. Let $A=U(x)\cap(P^*)^c(U(x)=u(x)\cup\{x\})$ so A is compact. Thus, we may pick $v\in A$ with $v\leqq A$. Since $x\leqq A$ and $x\notin A$, $x<v$. Furthermore,

$$v\in A\subseteq(P^*)^c\subseteq\{z\in C\mid a\leqq z\leqq b\}\cap P^c\subseteq P$$

so $V=u(x)\cap P$ is a neighborhood of v. We claim that v is the smallest element of V for the desired contradiction. For if $w\in V$ with $w<v$ then $w\in P$ and $a\leqq p_1<x<w<v\leqq b$ so $w\in P^*$. Then $w\in A$ contradicting the choice of v.

COROLLARY 4. *Let G be a locally connected or locally compact tl-group and let N be a closed prime convex l-subgroup of G. Then G/N is either discrete or has precisely the interval topology.*

We now show directly that the same result holds if G is locally convex.

THEOREM 5. *Let G be a locally convex tl-group and let N be a closed prime convex l-subgroup of G. Then G/N is either discrete or has precisely the interval topology.*

PROOF. Let U^* be a neighborhood of the coset N in G/N and let $U\pi\subseteq U^*$ where U is a neighborhood of the identity e in G, and π is the natural map. Let V be a convex neighborhood of e in G with $V\subseteq U$. Let $W\subseteq V$ be an open set with $W\vee W\subseteq W$ and $W\wedge W\subseteq W$. Then $C(W)$, the smallest convex set containing W, is (i) open [**1**, Lemma 1], (ii) contained in V, and (iii) a sublattice of G. Let $C=C(W)\cap(C(W))^{-1}$. Then C has these same three properties.

If $C\subseteq N$ then N is open so G/N is discrete. Otherwise, let $x\in C\backslash N$. We assume $Nx>N>Nx^{-1}$, the other case is similar. We claim that

$$\{Nz\mid Nx^{-1}<Nz<Nx\}\subseteq C\pi\subseteq V\pi\subseteq U\pi\subseteq U^*.$$

For pick Nz with say $N<Nz<Nx$. But since $e\leqq(z\vee e)\wedge(x\vee e)\leqq x\vee e\in C$, $(z\vee e)\wedge(x\vee e)\in C$ and so $Nz=N((z\vee e)\wedge(x\vee e))\in C\pi$. A similar argument works for the case $Nx^{-1}<Nz<N$.

3. **Further remarks.** It should be observed that every connected topological chain with more than one element is nonsectarian. For if there existed a neighborhood U of a nonendpoint $c \in C$ with say $c \leqq U$ then $U(c) = U \cup u(c)$ so $U(c)$ is open and closed in C. It follows that if C is a connected topological chain which is either locally compact, locally connected, or locally convex then C has exactly the interval topology. In fact, it is shown in [1] that in a connected topological chain the interval topology is equivalent to each of local compactness, local connectivity, and local convexity.

We now show that there does exist a totally ordered tl-group whose topology is neither the interval nor the discrete. Consider the real numbers as a vector space over the field of rational numbers. We may then consider the real numbers, as an additive group, to be the small direct sum of copies of the group of rational numbers. Let T be the topology for the real numbers whose open sets are generated by those sets of the form $I \cap C$ where I is a usual open interval in the real numbers and C is an open set from the Cartesian topology on the real numbers as the small direct sum of copies of the rationals; the interval topology on each copy of the rationals. Since the additive group of real numbers with this topology is clearly a topological group and since the topology contains as open sets at least the usual open sets, when the set of real numbers is ordered in the usual way we obtain a totally ordered tl-group. It is easy to see that the topology T lies properly between the interval and discrete topologies.

Finally, we recall that for any totally ordered set C which has the property that for all $x, y \in C$ there is an order-preserving permutation ϕ_{xy} of C with $x\phi_{xy} = y$, C is order isomorphic to $A(C)/C_z$ where $A(C)$ is the l-group of all order-preserving permutations of C [4] and C_z is the prime convex l-subgroup of order-preserving permutations fixing the arbitrary $z \in C$. Since $A(C)$ is completely distributive [5], $A(C)$ with the topology of α-convergence is a locally convex tl-group [7]. Since C_z is lattice-theoretically closed [5] and so closed in the topology of α-convergence [6], $A(C)/C_z$ with the projection topology has precisely the interval topology [7]. Thus, C with the interval topology is isomorphic as a topological chain to $A(C)/C_z$.

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UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37916

UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706