AMENABLE SUBSEMIGROUPS OF A LOCALLY COMPACT GROUP

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ABSTRACT. Let G be an amenable locally compact group and S an open subsemigroup of G. It is shown that S is amenable if, and only if, the right ideals of S have the finite intersection property.

Let S be a topological semigroup, i.e., S is a semigroup with a Hausdorff topology such that the mapping $(s, t) \rightarrow st$ of $S \times S$ into S is continuous. Let X be a closed subspace of $B(S)$, the space of all bounded real-valued functions on S equipped with the sup norm. Suppose also that X contains the constant functions. An element μ of X^* is a mean if $\|\mu\| = 1 = \mu(1_S)$, where $1_A(s) = 1$ for each s in A, and zero otherwise. For each s in S define the operator ℓ_s on $B(G)$ by $(\ell_s f)(s') = f(ss')$ for each f in $B(G)$ and s' in S. If X is invariant with respect to each ℓ_s, then ℓ_s^*, the adjoint of ℓ_s, maps X^* into itself. If this is the case, a mean in X^* is said to be left invariant (μ is a LIM) if $\ell_s^* \mu = \mu$ for each s in S.

A function f in $B(S)$ is left uniformly continuous if f is continuous and whenever $s_\alpha \rightarrow s_0$, s_α, s_0 in S; $\|1_{s_\alpha} f - 1_{s_0} f\| \rightarrow 0$. Let LUC$(S)$ denote the set of all left uniformly continuous functions on S. LUC(S) is a closed subspace of $B(S)$ (Namioka [**6**]). S is said to be amenable if there is a LIM in LUC$(S)^*$.

If S is a locally compact group then the space $L^\infty(S)$ of essentially bounded Borel measurable functions, defined as in Greenleaf, [**3**], has an important role. Means on $L^\infty(S)$ are defined relative to the ess sup norm and the operators ℓ_s, s in S, are defined similarly to the preceding. It is well known that if S is a locally compact group, the existence of a LIM in LUC$(S)^*$ is equivalent to the existence of a LIM in $L^\infty(S)^*$ (Greenleaf, [**3**]).

In [**2**], Frey proved that a subsemigroup S, of an amenable discrete group is amenable if, and only if, the right ideals of S have the finite intersection property. More recently, Wilde and Witz [**7**]

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proved a similar result for subsemigroup S of an amenable discrete semigroup with cancellation. In this paper we make use of some techniques of Wilde and Witz to prove the following theorem:

THEOREM A. *If G is an amenable locally compact group and S is an open subsemigroup of G, then the following are equivalent:*
(i) *S is amenable.*
(ii) *The open right ideals of S have the finite intersection property.*

It should be noted that condition (ii) of Theorem A is essential. Hochster [5] has presented an example of an amenable discrete group with a nonamenable subsemigroup.

The following ideas will be used repeatedly throughout this paper.

If $\mu \in \mathrm{LUC}(S)^*$ and $f \in \mathrm{LUC}(S)$, then the function $h(s)=\mu(\ell_s f)$ is in $\mathrm{LUC}(S)$ (Namioka [6]). Thus for μ, ν in $\mathrm{LUC}(S)^*$, the Arens product $\mu \Box \nu$, defined by $\mu \Box \nu(f)=\mu(h)$, where $h(s)=\nu(\ell_s f)$, is in $\mathrm{LUC}(S)^*$. Furthermore, the set of means on $\mathrm{LUC}(S)$, $M(S)$, is a semigroup with respect to the Arens product. If μ is a LIM and $\nu \in M(S)$ then $\nu \Box \mu=\mu$. (For this and other properties of the Arens product see Day [1].)

Let q be the evaluation map of S into $M(S)$, i.e. $q(s)(f)=f(s)$ for each s in S and f in $\mathrm{LUC}(S)$. q is a homomorphism and the convex hull of $q(S)$ is weak*-dense in $M(S)$. For $A \subset S$, $K(A)$ will denote the w^*-closed convex hull of $q(A)$.

Define the kernel of $M(S)$, $\ker(M(S))$, to be the smallest weak*-closed two-sided ideal of $M(S)$. Since $M(S)$ is weak*-compact, $\ker(M(S)) \neq \emptyset$. Also, if $L(S)$ is the set of LIM on $\mathrm{LUC}(S)$ then $L(S) \neq \emptyset$ if, and only if, $\ker(M(S))=L(S)$, and in this case $L(S)$ is the smallest closed right ideal of $M(S)$. (For this result see Witz [8].)

The following lemmas are required for proving Theorem A. The first is a generalization of a theorem of Day [1].

LEMMA 1. *Let G be a locally compact amenable group, and suppose that S is a Borel measurable subsemigroup of G. If there is a* LIM *μ on $L^\infty(G)$ such that $\mu(1_S)>0$ then S is amenable.*

PROOF. Let $\pi:\mathrm{LUC}(S) \to L^\infty(G)$ be the canonical embedding, i.e. $\pi(f)(g)=f(g)$ if $g \in S$ and zero otherwise. The fact that $\pi(f) \in L^\infty(G)$ is an immediate consequence of S being a Borel set. Let π^* be the adjoint of π, $\alpha=\mu(1_S)^{-1}$, and $\lambda=\alpha\pi^*(\mu)$. First note that $\lambda \in M(S)$ since $\lambda(1_S)=\alpha\pi^*(\mu)(1_S)=\alpha\mu(\pi(1_S))=\alpha\mu(1_S)=1$, and since λ is positive. Now, let f be in $\mathrm{LUC}(S)$, t be in S, and define $g(s) =\pi(\ell_t f)(s)-\ell_t(\pi f)(s)$ for each s in G. Then $g=g1_E$ where E

$=t^{-1}S\sim S$. One can easily show that for each t' in S, at most one of the elements $\{t^jt' : j=1, 2, \cdots\}$ is in E. Hence for each positive integer n,

$$\sum_{j=1}^{n} \ell_{t^j} 1_E(t') \leqq 1,$$

for each t' in S. Since E is measurable, $1_E \in L^\infty(G)$, and $n\mu(1_E) = \mu(\sum_{j=1}^{n} \ell_{t^j} 1_E) \leqq \mu(1_S) < \infty$. Thus $\mu(1_E)=0$, and so $|\mu(g)| = |\mu(g1_E)| \leqq \mu(\|g\| 1_E) = 0$. Therefore, for each f in LUC(S) and t in S,

$$\lambda(\ell_s f) = \alpha\pi^*\mu(\ell_s f) = \alpha\mu(\pi(\ell_s f)) = \alpha\mu(\ell_s(\pi f)) = \alpha\mu(\pi f) = \lambda(f).$$

Hence, $\lambda \in L(S)$.

LEMMA 2. *Let G be a locally compact amenable group, and S an open subsemigroup of G. If there is a* LIM *μ on* LUC(G) *such that μ is in $K(S)$, then S is amenable.*

PROOF. Suppose such a μ is given. Let U be a compact subset of S^{-1} such that the left Haar measure of U, $|U|$, is positive. Let ϕ_U be the normalized (relative to the L^1-norm) characteristic function of U. For each f in $L^\infty(G)$ the convolutions $\phi_U * f$ and $f * \phi_U$ are defined by

$$\phi_U * f(s) = \int \phi_U(t) f(t^{-1}s)\,dt$$

and

$$f * \phi_U(s) = \int f(t)\phi_U(t^{-1}s)\,dt$$

where dt is the left Haar measure on G. (Note that the definition of $f*\phi_U$ is dependent on the compactness of U.) It is easily seen that for each f in $L^\infty(G)$, $\phi_U * f * \phi_U$ is uniformly continuous (both right and left). By restriction, μ is a LIM on the bounded uniformly continuous functions. By a standard argument (see Greenleaf [3, p. 28]), if $\bar{\mu}(f) = \mu(\phi_U * f * \phi_U)$ for each f in $L^\infty(G)$, then $\alpha\bar{\mu}$ is a LIM on $L^\infty(G)$, where $\alpha = |U| / |U^{-1}|$.

Now, if $s \in S$, then

$$1_S * \phi_U(s) = \int 1_S(t)\phi_U(t^{-1}s)\,dt = \int_S \phi_U(t^{-1}s)\,dt = |U|^{-1} |sU^{-1} \cap S|.$$

Hence

$$[\phi_U*(1_S*\phi_U)](s) = \int \phi_U(t)(\,|U|^{-1}|t^{-1}sU^{-1}\cap S|\,)dt$$

$$= |U|^{-2}\int_U |t^{-1}sU^{-1}\cap S|\,dt.$$

Since $U^{-1}\subset S$, $|t^{-1}sU^{-1}\cap S| = |U^{-1}|$ for each t in U. Thus $\phi_U*f*\phi_U(s) = \alpha^{-1}$ for each s in S.

Finally, since $\mu\in K(S)$, $\bar{\mu}(1_S) = \alpha\mu(\phi_U*f*\phi_U) = 1$. An application of Lemma 1 establishes the result.

We are now able to prove Theorem A.

PROOF (THEOREM A). (ii)⇒(i). Let H be the subgroup of G generated by S. H is open and hence amenable. In [7], Wilde and Witz show that for each h in H, $hS\cap S$ is a right ideal of S. Since $hS\cap S$ is open for each h in H, $\{hS\cap S: h\in H\}$ has the finite intersection property; hence also $\{q(hS\cap S)^-: h\in H\}$ where A^- denotes the w^*-closure of any set $A\subset m^*(H)$. Set $E = \bigcap\{q(hS\cap S)^-: h\in H\}$. By the w^*-compactness of $M(H)$, $E\neq\emptyset$, and $E\subseteq q(S)^-$. One can easily show that E is a left ideal of $q(H)^-$, (with the Arens product), and hence, $K(E)$ is a left ideal of $M(H)$. Therefore, $K(E)\cap L(H) = K(E)\cap\ker(M(H)) \supset \ker(M(H))\boxdot K(E)\neq\emptyset$. Lemma 2 now applies.

(i)⇒(ii). Suppose I, J are open disjoint right ideals of S. Define Φ_U as in the proof of Lemma 2. Consider the functions 1_I, 1_J as elements of $L^\infty(G)$. Then

$$1_I * \Phi_{\tilde{U}}(s) = \int 1_I(t)\Phi_{\tilde{U}}(t^{-1}s)dt = \int_I \Phi_U(s^{-1}t)dt = |U|^{-1}|sU\cap I| = 1$$

for each s in I. Similarly, $1_J*\Phi_{\tilde{U}}(s) = 1$ for each s in J. Also, $(1_J*\Phi_{\tilde{U}})_{|S}$ and $(1_I*\Phi_{\tilde{U}})_{|S}$ are in LUC(S), and $[(1_I+1_J)*\Phi_{\tilde{U}}]_{|S} \leqq (1_S*\Phi_{\tilde{U}})_{|S} = 1_S$. Hence, if $\mu\in K(I)\cap K(J)$ then

$$1 = \mu(1_S) \geqq \mu([1_I*\Phi_{\tilde{U}}]_{|S}) + \mu([1_J*\Phi_{\tilde{U}}]_{|S}) = 2.$$

Thus $K(I)\cap K(J) = \emptyset$ if $I\cap J = \emptyset$. But, since S is amenable, $L(S) = \ker(M(S))$ is the smallest closed right ideal of $M(S)$, and thus $K(I)\cap K(J)\supset L(S)\neq\emptyset$. This contradiction shows that (i)⇒(ii).

REFERENCES

1. M. M. Day, *Amenable semigroups*, Illinois J. Math. 1 (1957), 509–544, MR 19, 1067.

2. A. H. Frey, *Studies in amenable semigroups*, Thesis, Univ. of Washington, Seattle, 1960.

3. F. P. Greenleaf, *Invariant means of topological groups*, Van Nostrand, Princeton, N. J., 1969.

4. E. Hewitt and K. Ross, *Abstract harmonic analysis*. Vol. I: *Structure of topological groups. Integration theory, group representations*, Die Grundlehren der math. Wissenschaften, Band 115, Academic Press, New York, and Springer-Verlag, Berlin and New York, 1963. MR **28** #158.

5. M. Hochster, *Subsemigroups of amenable groups*, Proc. Amer. Math. Soc. **21** (1969), 363–364. MR **39** #1575.

6. I. Namioka, *On certain actions of semi-groups on L-spaces*, Studia Math., **29** (1967), 63–77. MR **36** #6910.

7. C. Wilde and K. Witz, *Invariant means and the Stone-Čech compactification*, Pacific J. Math. **21** (1967), 577–586. MR **35** #3423.

8. K. G. Witz, *Applications of a compactification for bounded operator semigroups*, Illinois J. Math. **8** (1964), 685–696. MR **31** #2626.

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