REMARKS ON PSEUDO-CONTRACTIVE MAPPINGS

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ABSTRACT. Let X be a Banach space, $D\subset X$. A mapping $U:D\to X$ is said to be pseudo-contractive if for all $u,v\in D$ and all $r>0$, $\|u-v\|\leqq\|(1+r)(u-v)-r(U(u)-U(v))\|$. This concept is due to F. E. Browder, who showed that $U:X\to X$ is pseudo-contractive if and only if $I-U$ is accretive. In this paper it is shown that if X is a uniformly convex Banach, B a closed ball in X, and U a Lipschitzian pseudo-contractive mapping of B into X which maps the boundary of B into B, then U has a fixed point in B. This result is closely related to a recent theorem of Browder.

Let X be a Banach space and $D\subset X$. A mapping $U:D\to X$ is said to be *pseudo-contractive* (Browder [4]) if for all u, $v\in D$ and all $r>0$,

$$\|u-v\| \leqq \|(1+r)(u-v)-r(U(u)-U(v))\|.$$

This class of mappings is easily seen to be more general than the class of nonexpansive mappings; that is, mappings U for which

$$\|U(x)-U(y)\| \leqq \|x-y\|, \qquad x, y\in D.$$

However, the main interest in pseudo-contractive mappings stems from the firm connection which exists between these mappings and the important class of accretive mappings; namely, U is pseudo-contractive if and only if $I-U$ is accretive [4, Proposition 1]. Thus the mapping theory for accretive mappings is closely related to fixed-point theory of pseudo-contractive mappings. Using highly analytic techniques, and relying on this connection, Browder has proved the following theorem.

THEOREM 1 [4]. *Let X be a uniformly convex Banach space, B a closed ball in X, G an open set containing B. Let U be a pseudo-contractive mapping of G into X such that U maps the boundary of B into B. Suppose also that U is demicontinuous and that either* (a) *U is uniformly continuous in the strong topology on bounded subsets of X, or* (b) *X^* is uniformly convex. Then U has a fixed point in B.*

The object of this note is to give an elementary geometric proof of a theorem which is a slight variation of the "(a) version" of the above.

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We strengthen the assumption of demicontinuity of U and condition (a) by simply requiring U to be Lipschitzian, but at the same time it is not necessary for us to assume that U is defined on an open set containing B. (Also see Remark 2 below.)

THEOREM 2. *Let X be a uniformly convex Banach space and B a closed ball in X. Let U be a Lipschitzian pseudo-contractive mapping of B into X such that U also maps the boundary of B into B. Then U has a fixed point in B.*

PROOF. We may assume without loss of generality that B is a ball centered at the origin with radius ρ. Let ∂B denote the boundary of B. For each $r>0$, u, $v \in B$,

$$\|u - v\| \leqq \|(1 + r)(u - v) - r(U(u) - U(v))\|. \tag{1}$$

Letting $\lambda = r/(1+r)$, (1) is equivalent to

$$(1 - \lambda)\|u - v\| \leqq \|(u - v) - \lambda(U(u) - U(v))\|, \qquad \lambda > 0. \tag{2}$$

Let $T_\lambda = I - \lambda U$. Then (2) implies

$$\|T_\lambda(u) - T_\lambda(v)\| \geqq (1 - \lambda)\|u - v\|, \qquad u, v \in B. \tag{3}$$

Since U is Lipschitzian, there is a constant M such that

$$\|U(u) - U(v)\| \leqq M\|u - v\|.$$

Select $\lambda > 0$ so that $\lambda M < 1$ and $\lambda < 1$, and let $U_\lambda = \lambda U$. Then

$$\|U_\lambda(u) - U_\lambda(v)\| = \lambda\|U(u) - U(v)\| \leqq \lambda M\|u - v\| \tag{4}$$

so U_λ is strictly contractive on B. Also, since $\|U(x)\| \leqq \rho$ if $x \in \partial B$, $\|U_\lambda(x)\| \leqq \lambda\rho$ for $x \in \partial B$. Let

$$y^* \in B_1 = \{x \in X : \|x\| \leqq (1 - \lambda)\rho\}.$$

Define $\overline{U}_\lambda$ as follows: For $x \in B$, let $\overline{U}_\lambda(x) = U_\lambda(x) + y^*$. Then if $x \in \partial B$,

$$\|\overline{U}_\lambda(x)\| \leqq \|U_\lambda(x)\| + \|y^*\| \leqq \lambda\rho + (1 - \lambda)\rho = \rho,$$

so $\overline{U}_\lambda$ maps the boundary of B into B. This fact may be used to easily show that $F = (I + \overline{U}_\lambda)/2$ maps B into B. Also, since $\|\overline{U}_\lambda(x) - \overline{U}_\lambda(y)\| = \|U_\lambda(x) - U_\lambda(y)\|$, (4) implies that $\overline{U}_\lambda$ is strictly contractive. Thus F is also strictly contractive and application of the Banach Contraction Principle to F yields a point $x^* \in B$ such that $F(x^*) = x^* = \overline{U}_\lambda(x^*)$. Hence $\lambda U(x^*) + y^* = x^*$. Since $\lambda U = I - T_\lambda$, we have $x^* - T_\lambda(x^*) + y^* = x^*$ so $T_\lambda(x^*) = y^*$. Thus we have proved

$$T_\lambda[B] \supset B_1; \qquad T_\lambda^{-1}[B_1] \subset B.$$

Therefore $(1-\lambda)T_\lambda^{-1}: B_1 \to B_1$. By (3), $(1-\lambda)T_\lambda^{-1}$ is nonexpansive and so by the theorem of Kirk [5] (Browder [3]), $(1-\lambda)T_\lambda^{-1}$ has a fixed point $z \in B_1$. Thus, letting $z' = z/(1-\lambda)$, $T_\lambda(z') = z$ from which $z' - \lambda U(z') = z = (1-\lambda)z'$, yielding $U(z') = z'$.

REMARK 1. The assumptions on the space X may be weakened in both Theorems 1 and 2. It is only necessary that X be reflexive and B possess "normal structure" [2]. (If the norm of X is not strictly convex the possibly stronger assumption of "complete normal structure" [1] is necessary in Theorem 1.)

REMARK 2. In Theorem 2 it is only necessary to assume that U satisfies inequality (1) for *some* r, $0 < r < 1$, for which U has Lipschitz norm less than $(r+1)/r$.

Because its proof is almost identical with the one just given, we include a theorem for accretive mappings which may be of independent interest.

Let (x, w) denote the pairing of an element x of X and the element w of the conjugate space X^*. Define $J(x) = \{w \in X^*: (x, w) = \|x\|^2, \|w\| = \|x\|\}$.

DEFINITION [4]. A mapping $T: D \to X$ is said to be *accretive* if for all $u, v \in D$, $(T(u) - T(v), w) \geqq 0$, $w \in J(u-v)$.

THEOREM 3. *Let X be a Banach space and B a closed ball in X centered at the origin. Let T be an accretive mapping of B into X and suppose T is also Lipschitzian. If T maps the boundary of B into B then there is an element $x \in B$ such that $x + T(x) = 0$.*

PROOF. By [4, Proposition 1], $U = I - T$ is pseudo-contractive. Let

$$T_r = (1 + r)I - rU, \qquad r > 0,$$

and apply inequality (1) of the preceding argument to obtain

$$\|T_r(u) - T_r(v)\| \geqq \|u - v\| \qquad (u, v \in B).$$

Since T is Lipschitzian, one may choose $r > 0$ small enough that $F_r = -rT$ is strictly contractive. Assume $r < 1$ and let

$$y^* \in B_1 = \{x \in X: \|x\| \leqq (1 - r)\rho\}$$

(where ρ is the radius of B). As before, the mapping $\overline{F}_r$ defined by $F_r(x) + y^*$, $x \in B$, is strictly contractive on B mapping the boundary of B into B. Thus for some $x^* \in B$, $\overline{F}_r(x^*) = x^* = F_r(x^*) + y^*$. Since $F_r = -rT = I - T_r$, we conclude $T_r(x^*) = y^*$ yielding $T_r[B] \supset B_1$. From this, $(1-r)T_r^{-1}: B_1 \to B_1$. But $(1-r)T_r^{-1}$ is strictly contractive because T_r^{-1} is nonexpansive (see above). Application of the Contraction

Principle to $(1-r)T_r^{-1}$ yields a point $z \in B_1$ such that $(1-r)T_r^{-1}(z)=z$. Letting $z'=z/(1-r)$ it follows that $z=T_r(z')=(1+r)z'-rU(z')$ which implies $U(z')=2z'$. Since $U=I-T$, $z'+T(z')=0$, completing the proof.

In contrast to Theorem 2, no assumptions on the Banach space X are necessary in the above theorem. This is because $(1-r)T_r^{-1}$ is strictly contractive in the above argument, whereas at the analogous step in the proof of Theorem 2, $(1-\lambda)T_\lambda^{-1}$ is only nonexpansive.

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