FREE DIFFERENTIABLE ACTIONS OF S^1 AND S^3 ON HOMOTOPY SPHERES

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ABSTRACT. It is shown that there are homotopy $(4n+1)$- or $(4n+3)$-spheres which admit infinitely many differentiable free actions of S^1 or S^3 with characteristic homotopy spheres in certain dimensions and without characteristic homotopy spheres in some other dimensions.

1. Preliminaries. Throughout this paper, Z denotes the ring of integers, Q the field of rational numbers, $\sigma(M)$ the index of the smooth manifold M, and $\tau(M)$ the tangent bundle of M. We let CP^n be the complex projective n-space and QP^n the quaternion projective n-space, all having the usual differentiable structure.

Let (S^i, Σ^m), $i=1$ or 3, be a free differentiable action of S^i on the homotopy m-sphere Σ^m, $m=2n+1$ or $4n+3$, with orbit space $N=\Sigma^m/S^i$ and $f:N\to CP^n$ (respectively, $f:N\to QP^n$) a homotopy equivalence which is transverse regular on the submanifold CP^{n-k} (resp. QP^{n-k}) with $n-k>2$ (resp. $n-k>1$). By a *characteristic homotopy q-sphere* Σ^q of Σ^m, we mean a homotopy q-sphere Σ^q which is S^i-invariant with $\Sigma^q/S^i=M$, and such that $f|M$ is a homotopy equivalence, where $M=f^{-1}(CP^{n-k})$, $q=2n+1-2k$; or $M=f^{-1}(QP^{n-k})$, $q=4n+3-4k$. We shall assume that the dimension of M is divisible by 4. Then there is an obstruction to make f normally cobordant to $f':N\to CP^n$ (resp. $f':N\to QP^n$) a homotopy equivalence, such that if $M'=(f')^{-1}(CP^{n-k})$ (resp. $(f')^{-1}(QP^{n-k})$), then $f':(N, M')\to(CP^n, CP^{n-k})$ (resp. $f':(N, M')\to(QP^n, QP^{n-k})$) is a homotopy equivalence on each term [2, 2.14]. The obstruction lies in the group $8Z$, and we shall denote it by $I_{2k}(S^1, \Sigma^{2n+1})$ (resp. $I_{4k}(S^3, \Sigma^{4n+3})$). Now we restate a result of Browder in [2, 4.4] as follows:

1.1. THEOREM [2]. *Let (S^1, Σ^{2n+1}) or (S^3, Σ^{4n+3}) be a free differentiable action. Then*

$$I_{2k}(S^1, \Sigma^{2n+1}) = \langle L_q(\tau(N) \oplus k\rho^{-1}), \chi_{n-k}\rangle - 1,$$
$$I_{4k}(S^3, \Sigma^{4n+3}) = \langle L_q(\tau(N) \oplus k\rho^{-1}), \chi_{n-k}\rangle - \sigma(QP^{n-k}),$$

where ρ is the canonical bundle over $N=\Sigma^{2n+1}/S^1$ or Σ^{4n+3}/S^3, associated

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to the principal bundle $S^1 \to \Sigma^{2n+1} \to \Sigma^{2n+1}/S^1$ *or* $S^3 \to \Sigma^{4n+3} \to \Sigma^{4n+3}/S^3$ *respectively,* ρ^{-1} *the additive inverse of* ρ, χ_{n-k} *is the generator of* $H_{4q}(N)$, $n-k=2q$ *or* $n-k=q$ $(q>1)$, *and* L_q *the qth component of Hirzebruch's L-genus* [3],

$$L: (KO)^{\sim}(N) \to \sum_{i \geq 0} H^{4i}(N; Q).$$

In particular, the action has a characteristic homotopy $(2n+1-2k)$-*sphere (resp.* $(4n+3-4k)$-*sphere) if and only if* $\langle L_q(\tau(N) \oplus k\rho^{-1}), \chi_{n-k}\rangle = 1$ *or* $\langle L_q(\tau(N) \oplus k\rho^{-1}), \chi_{n-k}\rangle = \sigma(QP^{n-k})$ *respectively.*

In [6], we compared the invariants $I_2(S^1, \Sigma^{4n+3})$ and $I_4(S^3, \Sigma^{4n+3})$ for any free differentiable action of S^3 on homotopy sphere Σ^{4n+3}, where $S^1 \subset S^3$, and proved the following theorem:

1.2 THEOREM [6]. *Suppose that a free differentiable action of* S^3 *on a homotopy* $(4n+3)$-*sphere* Σ^{4n+3} *is given,* $n \geqq 3$, *and let* S^1 *be the subgroup of* S^3. *Then* $I_2(S^1, \Sigma^{4n+3}) = I_4(S^3, \Sigma^{4n+3})$.

2. Main theorems.

2.1. LEMMA [4]. *Let M denote the manifold* CP^{2n} *or* QP^n *for* $n \geqq 2$, *and* Δ *the subgroup of* $(KO)^{\sim}(M)$ *consisting of all* ξ *with rational Pontrjagin classes* $p_1(\xi)=0, \cdots, p_{[n/2]}(\xi)=0$. *Then the map*

$$\bar{L}: \Delta \to \sum_{i>[n/2]} H^{4i}(M; Q)$$

defined by

$$\bar{L}(\xi) = L_{[n/2]+1}(\xi) + L_{[n/2]+2}(\xi) + \cdots$$

for $\xi \in \Delta$ *is linear, and induces an isomorphism*

$$\bar{L}: \Delta \otimes Q \to \sum_{i>[n/2]} H^{4i}(M; Q).$$

2.2. THEOREM. *There are infinitely many differentiably distinct free actions of* S^1 *(resp.* S^3*) on homotopy* $(4n+1)$-*spheres (resp.* $(4n+3)$-*spheres) with characteristic homotopy* $(4n+1-4k)$-*spheres (resp.* $(4n+3-4k)$-*spheres) for* $n \geqq 5$ *and* $n-k \geqq 2$.

PROOF. Let M denote the manifold CP^{2n} or QP^n and $\tau \in (KO)^{\sim}(M)$ the stable tangent bundle of M. Let

$$J: (KO)^{\sim}(M) \to J(M)$$

be the J-homomorphism [1]. We shall show that there are infinitely many $\xi \in (KO)^{\sim}(M)$ with different rational Pontrjagin classes satisfying the following equations:

$$L_n(\xi) + L_{n-1}(\xi)\cdot L_1(\tau) + \cdots + L_1(\xi)\cdot L_{n-1}(\tau) = 0, \tag{1}$$

$$J(\xi) = 0, \tag{2}$$

$$L_q(\xi) + L_{q-1}(\xi)\cdot L_1(\tau \oplus s\rho^{-1}) + \cdots + L_1(\xi)\cdot L_{q-1}(\tau \oplus s\rho^{-1}) = 0, \tag{3}$$

where $q = n-k$, ρ is as in Theorem 1.1, and $s=2k$ in the S^1 case, $s=k$ in the S^3 case, $\cdot$ denotes the cup product in $H^*(M; Q)$.

The solutions of the following system of linear equations (4) form a vector space V over Q of dimension greater than or equal to $n-[n/2]-2$:

$$\begin{aligned} L_{n-1}(\tau)\cdot x_1 + \cdots + L_1(\tau)\cdot x_{n-1} + x_n &= 0, \\ L_{q-1}(\tau \oplus s\rho^{-1})\cdot x_1 + \cdots + L_1(\tau \oplus s\rho^{-1})\cdot x_{q-1} + x_q &= 0, \\ x_1 = 0, \quad x_2 = 0, \cdots, x_{[n/2]} &= 0, \end{aligned} \tag{4}$$

where $x_i \in H^{4i}(M; Q)$. Since $n \geqq 5$, hence $n-[n/2]-2 \geqq 1$, and so $\dim V \geqq 1$. According to Lemma 2.1, we have an isomorphism

$$\bar{L}: \Delta \otimes Q \xrightarrow{\approx} \sum_{i>[n/2]} H^{4i}(M; Q)$$

and we may regard V as a subspace of $\sum_{i>[n/2]} H^{4i}(M; Q)$. Thus there exists a nonzero vector v in V and an element of infinite order ξ in Δ such that $\bar{L}(\xi) = v$. Since $J(M)$ is finite, there exists an integer $t \neq 0$ such that $J(t\xi) = tJ(\xi) = 0$. Therefore the elements of the infinite subgroup of $(KO)^{\sim}(M)$ generated by $t\xi$ satisfy (1), (2) and (3) with different Pontrjagin classes. Now given a bundle ξ over M satisfying (1) and (2), there is a homotopy equivalence $f: N \to M$ such that the stable tangent bundle of N is $f^*(\xi \oplus \tau)$ [2, 2.10]. Hence there are infinitely many smooth manifolds N of the same homotopy type of M with stable tangent bundles of the form $f^*(\xi \oplus \tau)$ (cf. [4]).

We can easily construct a standard action of S^1 or S^3 on S^{4n+1} or S^{4n+3} with characteristic $(4n+1-4k)$-sphere $S^{4n+1-4k}$ or $(4n+3-4k)$-sphere $S^{4n+3-4k}$, hence $\langle L_q(\tau \oplus s\rho^{-1}), \chi_q\rangle = 1$ or $\sigma(QP^{n-k})$ according to the action of S^1 or S^3 by 1.1. We see that

$$\begin{aligned} I_{4k}(S^1, \Sigma^{4n+1}) &= \langle L_q(\xi \oplus \tau \oplus s\rho^{-1}), \chi_q\rangle - 1 \\ &= \langle L_q(\xi) + L_{q-1}(\xi)L_1(\tau \oplus s\rho^{-1}) + \cdots \\ &\qquad + L_1(\xi)L_{q-1}(\tau \oplus s\rho^{-1}), \chi_q\rangle \\ &\quad + \langle L_q(\tau \oplus s\rho^{-1}), \chi_q\rangle - 1 = 0 \quad \text{by (3).} \end{aligned}$$

Similarly, $I_{4k}(S^3, \Sigma^{4n+3})=0$. The results then follow from Theorem 1.1.

2.3. COROLLARY. *There are infinitely many free differentiable actions of S^1 on homotopy $(4n+3)$-spheres with codimension 2 characteristic homotopy spheres.*

PROOF. Let S^3 act freely and differentiably on homotopy $(4n+3)$-sphere Σ^{4n+3}. Then we have fibration

$$\eta: S^2 \to \Sigma^{4n+3}/S^1 \xrightarrow{\pi} \Sigma^{4n+3}/S^3$$

with total Pontrjagin class $P(\Sigma^{4n+3}/S^1)=\pi^*(P(\Sigma^{4n+3}/S^3)P(\eta))$. Thus we have infinitely many differentiable actions of S^3 on homotopy spheres Σ^{4n+3} with codimension 4 characteristic homotopy spheres such that the orbit spaces of the restricted S^1 actions still have different rational Pontrjagin classes by Theorem 2.2. We apply Theorem 1.2 to get $I_2(S^1, \Sigma^{4n+3})=I_4(S^3, \Sigma^{4n+3})=0$. Hence the conclusion follows from Theorem 1.1.

We remark that Theorem 2.2 can be generalized to the actions with various different characteristic homotopy spheres of distinct dimensions.

2.4. THEOREM. *There are infinitely many differentiably distinct free actions of S^1 (resp. S^3) on homotopy $(4n+1)$-spheres (resp. $(4n+3)$-spheres) with characteristic homotopy spheres in dimensions $4n+1-4k_i$, $i=1, \cdots, j$ (resp. $4n+3-4k_i$, $i=1, \cdots, j$) for $n \geqq 3+2j$, $k_1>k_2> \cdots >k_j \geqq 1$ and $n-k_1 \geqq 2$.*

For the proof we replace (3) and (4) in the proof of Theorem 2.2 by (3′) and (4′) and applying 1.1 for various dimensions.

$$(3') \quad L_{q_i}(\xi)+L_{q_i-1}(\xi)L_1(\tau\oplus s_i\rho^{-1})+ \cdots +L_1(\xi)L_{q_i-1}(\tau\oplus s_i\rho^{-1})=0$$

for $i=1, \cdots, j$; where $q_i=n-k_i$, $s_i=2k_i$ or k_i depending on the action of S^1 or S^3 respectively.

$$(4') \quad \begin{aligned} L_{n-1}(\tau)\cdot x_1 + \cdots + L_1(\tau)\cdot x_{n-1} + x_n &= 0, \\ L_{q_i-1}(\tau \oplus s_i\rho^{-1})\cdot x_1 + \cdots + L_1(\tau \oplus s_i\rho^{-1})\cdot x_{q_i-1} + x_{q_i} &= 0, \\ i &= 1, \cdots, j, \\ x_1 = 0, \quad x_2 = 0, \cdots, x_{[n/2]} &= 0. \end{aligned}$$

If we replace (3) by

$$L_{q-1}(\xi)\cdot L_1(\tau \oplus s\rho^{-1}) + \cdots + L_1(\xi)\cdot L_{q-1}(\tau \oplus s\rho^{-1}) = 0$$

in Theorem 2.2, we have infinitely many differentiably distinct free actions of S^1 or S^3 on Σ^{4n+1} or Σ^{4n+3} with $I_{4k}(S^i, \Sigma^m) = \langle L_q(\xi), \chi_q\rangle$, $i=1, 3$; $m=4n+1$ or $4n+3$. We claim that there are infinitely many such $\xi \in (KO)^{\sim}(M)$ with $\langle L_q(\xi), \chi_q\rangle \neq 0$.

2.5. THEOREM. *There are infinitely many free differentiable S^3-actions (resp. S^1-actions) on homotopy $(4n+3)$-spheres (resp. $(4n+1)$-spheres), $n \geqq 5$, so that none of them has a characteristic homotopy $(4q+3)$-sphere (resp. $(4q+1)$-sphere), $[n/2]+1 \leqq q \leqq n-1$.*

PROOF. The proof is exactly the same as Theorem 2.2. Instead of (1), (2), (3), we consider

$$(1'') \qquad L_n(\xi) + L_{n-1}(\xi)\cdot L_1(\tau) + \cdots + L_1(\xi)\cdot L_{n-1}(\tau) = 0,$$

$$(2'') \qquad J(\xi) = 0,$$

$$(3'') \qquad L_{q-1}(\xi)\cdot L_1(\tau \oplus s\rho^{-1}) + \cdots + L_1(\xi)\cdot L_{q-1}(\tau \oplus s\rho^{-1}) = 0.$$

We shall have a system of linear equations corresponding to $(1'')$ and $(3'')$. The space V of solutions is exactly of dimension $n-[n/2]-1$ if $q=[n/2]+1$ and is $n-[n/2]-2$ if $q \geqq [n/2]+2$ because $L_1(\xi)=0, \cdots, L_{[n/2]}(\xi)=0$. We claim that $L_q(\xi)\neq 0$ for infinitely many $\xi \in \Delta$ satisfying $(1'')$, $(2'')$ and $(3'')$. To prove this it suffices to show that there is $\xi \in \bar{L}^{-1}(V)$ such that $L_q(\xi)\neq 0$ (see proof of 2.2). Suppose not, then $L_q(\xi)=0$ for all $\xi \in \bar{L}^{-1}(V)$. Thus V is the space of solutions of the following linear equations:

$$L_{n-1}(\tau)\cdot x_1 + \cdots + L_1(\tau)\cdot x_{n-1} + x_n = 0,$$

$$L_{q-1}(\tau \oplus s\rho^{-1})\cdot x_1 + \cdots + L_1(\tau \oplus s\rho^{-1})\cdot x_{q-1} = 0,$$

$$x_1 = 0, \cdots, x_{[n/2]} = 0, \quad x_q = 0.$$

Hence $\dim V = n-[n/2]-2$ if $q=[n/2]+1$, and $\dim V \leqq n-[n/2]-3$ if $q \geqq [n/2]+2$. This is a contradiction. Therefore we have infinitely many $\xi \in (KO)^{\sim}(M)$, $M=QP^n$ or CP^{2n} satisfying $(1'')$, $(2'')$, $(3'')$ and $L_q(\xi)\neq 0$. Let (S^3, Σ^{4n+3}) (resp. (S^1, Σ^{4n+1})) be the free differentiable action of S^3 (resp. S^1) constructed from ξ as in Theorem 2.2. Then

$$I_{4n-4q}(S^3, \Sigma^{4n+3}) = \langle L_q(\xi), \chi_q\rangle \neq 0,$$

or

$$I_{4n-4q}(S^1, \Sigma^{4n+1}) = \langle L_q(\xi), \chi_q\rangle \neq 0.$$

This completes the proof of the theorem.

As an easy consequence of Theorem 2.5, Theorem 1.1 and Theorem 1.2, we have

2.6. COROLLARY. *There are infinitely many free differentiable S^1-actions on homotopy* $(4n+3)$*-spheres, $n \geqq 5$, so that none of them has a codimension* 2 *characteristic homotopy sphere.*

Finally we remark that if we suppose n is large, then we can see from the proofs of Theorems 2.4 and 2.5 that there are infinitely many differentiable free S^1 or S^3-actions on homotopy $(4n+1)$-spheres or $(4n+3)$-spheres with characteristic homotopy spheres in certain dimensions and without characteristic homotopy spheres in some other dimensions.

REFERENCES

1. R. Bott, *Lectures on $K(X)$*, Harvard Univ., Cambridge, Mass., 1963.

2. W. Browder, *Surgery and the theory of differentiable transformation groups*, Proc. Conference on Transformation Groups, Springer-Verlag, Berlin and New York, 1968, pp. 1–46.

3. F. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie*, Ergebnisse der Mathematik und Grenzgebiete, Heft 9, Springer-Verlag, Berlin, 1956 English transl., Die Grundlehren der math. Wissenschaften, Band 131, Springer-Verlag, Berlin and New York, 1966. MR **18**, 509; MR **34** #2573.

4. W. C. Hsiang, *A note on free differentiable actions of S^1 and S^3 on homotopy spheres*, Ann. of Math. (2) **83** (1966), 266–272. MR **33** #731.

5. M. Kervaire and J. Milnor, *Groups of homotopy spheres*. I, Ann. of Math. (2) **77** (1963), 504–537. MR **26** #5584.

6. Hsu-Tung Ku and Mei-Chin Ku, *Characteristic spheres of free differentiable actions of S^1 and S^3 on homotopy spheres*, Univ. of Massachusetts, Amherst, Mass., 1969 (mimeographed).

7. D. Montgomery and C. T. Yang, *Free differentiable actions on homotopy spheres*, Proc. Conference on Transformation Groups, Springer-Verlag, Berlin and New York, 1968, pp. 175–192.

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