THE VALUE-SEMIGROUP OF A ONE-DIMENSIONAL GORENSTEIN RING

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In a conversation about [4], O. Zariski indicated to the author that there should be a relation between Gorenstein rings and symmetric value-semigroups, possibly allowing a new proof for a result of Herzog on complete intersections. In the following note it is shown that this is the case.

We use the following facts about Gorenstein rings: If R is a one-dimensional noetherian local ring with maximal ideal m and full ring of quotients $Q(R)$, then the following conditions are equivalent:

(a) R is a Gorenstein ring (by definition: m contains a nonzero divisor, which generates an irreducible ideal).

(b) Each principal ideal, generated by a nonzero divisor, is irreducible.

(c) The length of the R-module m^{-1}/R is 1.

(d) For each ideal a of R, which contains a nonzero divisor, $(a^{-1})^{-1}=a$. Here the inverse of an ideal is taken in $Q(R)$. For easy proofs of these equivalences see Berger [3].

Let $\overline{R}$ be the integral closure of R in $Q(R)$ and f the conductor from R to $\overline{R}$. If $\overline{R}$ is a finitely generated R-module, then f contains a nonzero divisor. If R is Gorenstein, then the length of the R-module $\overline{R}/f$ is $2d$, where d is the length of R/f. Roquette [5] gives the following simple proof: If

$$f = a_0 \subset \cdots \subset a_{d-1} \subset R$$

is a maximal chain of ideals in R, then

$$f = a_0 \subset \cdots \subset a_{d-1} \subset R \subset a_{d-1}^{-1} \subset \cdots \subset a_0^{-1} = f^{-1} = R$$

is a maximal chain of R-submodules of $\overline{R}$, because of $(a^{-1})^{-1}=a$, hence length $(\overline{R}/f)=2d$.

A local complete intersection is always a Gorenstein ring and by a theorem of Serre [6] each Gorenstein ring R, which has embedding dimension $\leqq \dim R+2$, is a complete intersection.

Now assume that R is analytically irreducible. This condition is equivalent with: $\overline{R}$ is a finitely generated R-module and a discrete valuation ring. Let $v: Q(R)\to Z$ be the corresponding valuation. We

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want to give a necessary and sufficient condition for R to be Gorenstein in terms of the value-semigroup $v(R)$ of R.

DEFINITION. A semigroup S of natural numbers (with $0 \in S$) is called *symmetric*,[1] if there is an integer n, such that the mapping $Z \to Z$, given by $z \to n-z$, maps elements of S onto nonelements and nonelements onto elements of S.

If S is a semigroup with 0, generated by natural numbers $n_1, \cdots, n_t$ with $(n_1, \cdots, n_t)=1$, then there is an integer $n \notin S$ such that $n+i \in S$ for $i=1, 2, \cdots$. For each $s \in S$ we have $n-s \notin S$. So the number of elements of S in $\{0, 1, \cdots, n\}$ is always less than or equal to the number of nonelements.

LEMMA. *S is symmetric iff in the set $\{0, 1, \cdots, n\}$ there are as many elements of S as nonelements.*

PROOF. If the condition is satisfied, then for $z \in \{0, 1, \cdots, n\}$, $z \notin S$ we must have $n-z \in S$, hence S is symmetric. Conversely, if S is symmetric, then the element n in the lemma equals the n in the definition of a symmetric semigroup. Since $z \to n-z$ maps $\{0, 1, \cdots, n\}$ onto itself, there are in this set as many elements of S as nonelements.

THEOREM. *Let R be a one-dimensional analytically irreducible noetherian local ring, $\overline{R}$ its integral closure in the quotient field K and $v: K \to Z$ the corresponding valuation. Assume R and $\overline{R}$ have the same residue class field (which is f.i. the case, if the residue class field of R is algebraically closed). Then R is Gorenstein iff the value-semigroup $v(R)$ is symmetric.*

PROOF. Choose n for $v(R)$ as above and let $c=n+1$. We claim: The conductor f from R to $\overline{R}$ is the set of all $x \in \overline{R}$ with $v(x) \geqq c$. Obviously f must be contained in this set. On the other hand, if $x \in \overline{R}$, $v(x) \geqq c$, then $v(x)=v(r)$ for some $r \in R$. Since R and $\overline{R}$ have the same residue class field, there is a unit e of R such that $v(x-e\cdot r)>v(x)$. By induction there is also an $r' \in R$ with $v(x-r') \geqq c'$, where c' is the least value of an element of f. All elements $y \in \overline{R}$ with $v(y) \geqq c'$ are in R, so $x \in R$. Now it is also clear that $x \in f$.

Assume that $v(R)$ is symmetric and $x \in m^{-1}$, $x \notin R$. If $v(x) \in v(R)$, we can find by a similar argument as above an $r \in R$ such that $v(x-r) \notin v(R)$. We still have $x-r \in m^{-1}$ and we may assume therefore that $v(x) \notin v(R)$. If $v(x)<n$, then, since $v(R)$ is symmetric, $n-v(x) \in v(R)$.

[1] Herzog [4] calls these semigroups "Sylvester-semigroups." For more details see his paper and also [1] and [2].

Choose $r_1 \in R$ with $v(r_1) = n - v(x)$. Then $v(r_1 x) = n$ and hence $r_1 x \notin R$, contradicting $x \in m^{-1}$. So $v(x) = n$ and m^{-1} contains besides R only elements of value n. This implies that m^{-1}/R is an R-module of length 1 and hence R is Gorenstein.

Assume now that R is Gorenstein and that $v_0 < v_1 < \cdots < v_{d-1}$ are those numbers in $\{0, 1, \cdots, n\}$, which are values of elements of R. Define a_i as the set of all elements $r \in R$ with $v(r) \geqq v_i$ $(i = 0, \cdots, d-1)$. Then

$$R = a_0 \supset a_1 \supset \cdots \supset a_{d-1} \supset f$$

is a strictly decreasing sequence of ideals of R. Moreover this sequence is maximal, because, if we adjoin to a_i an element $r \in R$ of value v_{i-1}, then we get all of a_{i-1}. It follows that $d = \text{length } (R/f)$. Since R is Gorenstein, we have $c = n + 1 = 2d$ and by the lemma $v(R)$ is symmetric.

COROLLARY (APÉRY [1]). *Let R be an analytically irreducible noetherian local ring of dimension* 1, *whose residue class field is algebraically closed and whose maximal ideal is generated by* 2 *elements. Then the value-semigroup of R is symmetric.*

In fact, R is a Gorenstein ring. For an easy direct proof see [3].

COROLLARY 2. *Let R be an analytically irreducible noetherian local ring of dimension* 1, *whose residue class field is algebraically closed and whose maximal ideal is generated by* 3 *elements. Then R is a complete intersection iff its value-semigroup is symmetric.*

By the theorem of Serre, mentioned above, Gorenstein ring and complete intersection here mean the same.

Corollary 2 gives a new proof for the local part of a result of Herzog [4], which states that an affine space-curve C with the parametric equations

$$x = t^a, \quad y = t^b, \quad z = t^c \quad (a, b, c \text{ natural numbers with } (a, b, c) = 1)$$

is idealtheoretically a complete intersection (globally), iff the semigroup S, generated by a, b, c is symmetric. In fact, S is the value-semigroup of the local ring R of C at the origin and R is analytically irreducible.

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