OPEN MAPPINGS AND THE LACK OF FULLY COMPLETENESS OF $\mathfrak{D}'(\Omega)$

CHARLES HARVEY AND F. REESE HARVEY

ABSTRACT. Consider a linear map T of one locally convex linear space into another which is densely defined and has a closed graph. We characterise the property that T is an open map in terms of two properties of its adjoint map T^*. These results are used to show that if Ω is an open subset of R_n for which there is a linear constant coefficient differential operator P such that Ω is P-convex but not strongly P-convex, then (i) $\mathfrak{D}'(\Omega)$ is not fully complete, (ii) the range of the adjoint map ${}^tP$ is closed but not bornological.

Introduction. In the following, a complex locally convex Hausdorff topological vector space will be referred to as a l.c.s. A linear map $T:E\to F$, whose domain D_T is a dense subspace of a l.c.s. E and whose range R_T is contained in a l.c.s. F such that T has a closed graph, will be called simply a *closed linear map*. The domain and range are assumed to have the relative topology. A closed linear map T is said to be *open* if $T(U\cap D_T)$ is an open subset of R_T for each open set U in E.

In §1, we give a characterization of an open map T in terms of properties of its adjoint map T^*. By placing restrictions on the class of spaces to which E and F belong, this result can be applied to derive several different functional analysis results; the Ptak open mapping theorem, a generalization of the result of C. Harvey [2] which relates the surjectivity of T to a priori estimates on T^*, and the closed range theorem.

In §2, the results of §1 are used to show that if P is a constant coefficient linear differential operator on $\mathfrak{D}'(\Omega)$ where Ω is a P-convex but not strongly P-convex open subset of $\boldsymbol{R}^n$, then the map $P:\mathfrak{D}'(\Omega)\to\mathfrak{D}'(\Omega)$ is open. Consequently, the quotient space $\mathfrak{D}'(\Omega)/P^{-1}(0)$ is not complete, which implies that $\mathfrak{D}'(\Omega)$ is not fully complete. Also, the range of the adjoint map ${}^tP:C_0^\infty\to C_0^\infty(\Omega)$, although closed, is not the inductive limit of metrizable l.c.s. (Equivalently, the range of ${}^tP$ is not bornological.)

1. **Open mappings.** A closed linear map $T:E\to F$ is said to be *weakly open* if it is open when E and F have the weak topologies

---

Received by the editors November 7, 1969.

AMS Subject Classifications. Primary 4601; Secondary 4640.

Key Words and Phrases. Locally convex linear space, open linear map, equicontinuous subset, fully complete space, Mackey space, $\mathfrak{D}'(\Omega)$, P-convex, strongly P-convex, bornological.

$\sigma(E, E')$ and $\sigma(F, F')$ respectively. The map T is called *nearly open* if for each neighborhood U of 0 in D_T, the closure of $T(U)$ in R_T is a neighborhood of 0 in R_T. If R_T is dense in F one can easily verify that the closure of $T(U)$ in F is a neighborhood of 0 in F.

A closed linear map $T:E \to F$ can be decomposed as $T = \psi \circ T_0 \circ \phi$ where $\phi: E \to E/T^{-1}(0)$ is a quotient map and $\psi: R_T \to F$ is an embedding. Since ϕ and ψ are open, weakly open, nearly open, and have closed graphs, one can easily verify that each of these properties is true for T if and only if it is true for T_0.

THEOREM 1.1. *Suppose $T:E \to F$ is a closed linear map. Then T is open if and only if T is weakly open and nearly open.*

PROOF. Suppose T is open. Then it is clear that T is nearly open. To prove that T is weakly open, we first observe that T_0 is open, hence T_0^{-1} is continuous. Since a continuous linear map is weakly continuous, T_0^{-1} is weakly continuous, so T_0 is weakly open.

Suppose next that T is nearly open and weakly open. Then the same is true of T_0. Consider a convex closed neighborhood U of 0 in $E/T^{-1}(0)$. Then $\mathrm{Cl}(T_0(U))$ is a convex neighborhood of 0 in R_T. But $T_0(U)$ is equal to its closure since T_0 is injective and weakly open. Thus T_0^{-1} is continuous which implies that T_0 is open.

The importance of the above theorem lies in the fact that both the concepts of weakly open and nearly open can be dualized. This is the content of the next lemma.

LEMMA 1.1. *Suppose that $T:E \to F$ is a closed linear map with a dense range. Then*

(i) *T is weakly open iff the range R_{T^*} of the adjoint map $T^*: F' \to E'$ is $\sigma(E', E)$ closed.*

(ii) *T is nearly open if and only if for each equicontinuous subset B of E', $(T^*)^{-1}(B)$ is an equicontinuous subset of F'.*

PROOF. For the proof of (i) see [1]. To prove (ii), we first observe that for any neighborhood U of 0 in E, $T(U \cap D_T)^0 = (T^*)^{-1}((U \cap D_T)^0)$. For if $f' \in T(U \cap D_T)^0$, then the map e' of e into (Te, f') is continuous. Thus $e' = T^*f'$, so $f' \in D_{T^*}$. Hence

$$\begin{aligned} T(U \cap D_T)^0 &= \{f' \in D_{T^*}: |(Te, f')| \leqq 1 \quad \text{for all } e \in U \cap D_T\} \\ &= \{f' \in D_{T^*}: |(Te, T^*f')| \leqq 1 \quad \text{for all } e \in U \cap D_T\} \\ &= (T^*)^{-1}((U \cap D_T)^0). \end{aligned}$$

Suppose T is nearly open. Consider an equicontinuous subset B of E'. $B \subset U^0 \subset (U \cap D_T)^0$ for some neighborhood U of 0 in E. Therefore, $(T^*)^{-1}(B) \subset T(U \cap D_T)^0$. But the closure of $T(U \cap D_T)$ is a neighborhood of 0 in F so $T(U \cap D_T)^0$ is an equicontinuous subset of F'. To prove the converse, it suffices to consider any closed absolutely convex absorbing neighborhood U of 0 in E. $(U \cap D_T)^0 = U^0$ is an equicontinuous subset of E', so $(T^*)^{-1}(U^0) = T(U \cap D_T)^0$ is an equicontinuous subset of F'. Thus $T(U \cap D_T)^{00}$ is a neighborhood of 0 in F. But the closure of $T(U \cap D_T)$ is $T(U \cap D_T)^{00}$ since $T(U \cap D_T)$ is absolutely convex. Thus T is nearly open.

We next combine Theorem 1.1 and Lemma 1.1 to obtain the following result.

THEOREM 1.2. *Suppose that $F: E \to F$ is a closed linear map with dense range. Then T is open if and only if both of the following conditions are satisfied*:

(a) *For each equicontinuous subset B of E', $(T^*)^{-1}(B)$ is an equicontinuous subset of F'.*

(b) *The range R_{T^*} of T^* is $\sigma(E', E)$ closed.*

By placing suitable restrictions on E or F one can either derive condition (b) from condition (a) or vice versa. First, we will show that if E is fully complete, then (a) implies (b). The result (Theorem 1.3) implies that if E and F are Fréchet spaces, then T is surjective if and only if certain a priori estimates hold for T^*, [2]. We also wish to observe that, since condition (a) is satisfied if and only if T is nearly open (part (ii) Lemma 1.1), the next theorem is equivalent to the Ptak open mapping theorem.

E is said to be *fully complete* if a subspace M of E' is $\sigma(E', E)$ closed whenever $M \cap U^0$ is $\sigma(E', E)$ closed for every neighborhood U of 0 in E.

THEOREM 1.3. *Suppose T is as described in Theorem* 1.2. *Assume that E is fully complete. Then condition* (a) *implies that T is open.*

PROOF. We must show that R_{T^*} is $\sigma(E', E)$ closed. Since E is fully complete, it suffices to show that $R_{T^*} \cap U^0$ is $\sigma(E', E)$ closed for every neighborhood U of 0 in E. Suppose $\{T^* f'_\alpha\}$ is a net in $R_{T^*} \cap U^0$ which is $\sigma(E', E)$ convergent to $e' \in E'$. Then $e' \in U^0$ since U^0 is $\sigma(E', E)$ closed. $(T^*)^{-1}(U^0)$ is equicontinuous by condition (b). Thus $(T^*)^{-1}(U^0)$ is relatively compact in the $\sigma(F', F)$ topology by the theorem of Alaoglu-Bourbaki. But each $f'_\alpha \in (T^*)^{-1}(U^0)$ so there is a subnet $\{f'_\beta\}$ which is $\sigma(F', F)$ convergent to an element $f' \in F'$. The mapping T^* has closed graph so it follows that $f' \in D_{T^*}$ and $T^* f' = e'$. Thus $e' \in R_{T^*}$. Hence $R_{T^*} \cap U^0$ is $\sigma(E', E)$ closed.

Next we show that if R_T is Mackey then condition (b) implies condition (a). This result (Theorem 1.4) generalizes the closed range theorem for Fréchet spaces since every subspace of a metrizable l.c.s. is a Mackey space.

THEOREM 1.4. *Suppose T is as described in Theorem 1.2. Assume that R_T is Mackey. Then condition (b) implies that T is open.*

PROOF. We must deduce condition (a). It is sufficient to consider the case where T is surjective. For consider an absolutely convex neighborhood V of 0 in F. Since R_T is dense in F, $V \subset \mathrm{Cl}(V \cap R_T)$. Thus $(V \cap R_T)^0 \subset V^0$. Consequently, each equicontinuous subset of R_T' is equicontinuous in F'.

Now suppose B is a $\sigma(E', E)$-closed equicontinuous set in E'. By the Alaoglu-Bourbaki Theorem, B is $\sigma(E', E)$ compact and hence $B \cap R_{T*}$ is $\sigma(E', E)$ compact since by hypothesis R_{T*} is $\sigma(E', E)$ closed. Since T is surjective, T^{*-1} is weakly continuous. Therefore $T^{*-1}(B)$ is $\sigma(F', F)$ compact. Since F is Mackey each set in F' which is $\sigma(F', F)$ compact is also equicontinuous. This proves that condition (b) implies condition (a).

2. **An incomplete quotient of $\mathfrak{D}'(\Omega)$.** Let Ω denote an open subset of $\boldsymbol{R}^n$, and P a linear differential operator with constant coefficients. Ω is said to be P-convex if $P: C^\infty(\Omega) \to C^\infty(\Omega)$ is surjective.

LEMMA 2.1. *If Ω is P-convex, then*
(i) ${}^tP: C_0^\infty(\Omega) \to C_0^\infty(\Omega)$ *has a bounded inverse and a closed range.*
(ii) $P: \mathfrak{D}'(\Omega) \to \mathfrak{D}'(\Omega)$ *is an open mapping.*

PROOF. For a proof that ${}^tP^{-1}(B)$ is bounded if B is bounded, see Hörmander [**3**, p. 150].

Suppose $\{\phi_\alpha\}$ is a net in $C_0^\infty(\Omega)$ with $\{{}^tP\phi_\alpha\}$ converging to $\psi \in C_0^\infty(\Omega)$. Then $\{{}^tP\phi_\alpha\}$ converges to ψ in $\mathcal{E}'(\Omega)$. The map $P: C^\infty(\Omega) \to C^\infty(\Omega)$ is by assumption surjective and hence open. Therefore the adjoint map from $\mathcal{E}'(\Omega)$ into $\mathcal{E}'(\Omega)$ has closed range by Theorem 1.2. Thus there exists an element $\phi \in \mathcal{E}'(\Omega)$ with ${}^tP\phi = \psi$. Since $\psi \in C_0^\infty(\Omega)$, ϕ is also an element of $C_0^\infty(\Omega)$ since $\phi = K * \psi$ where K is a fundamental solution of ${}^tP$.

An open subset Ω of $\boldsymbol{R}^n$ is said to be strongly P-convex if $P: \mathfrak{D}'(\Omega) \to \mathfrak{D}'(\Omega)$ is surjective. If Ω is strongly P-convex, then Ω is P-convex, but not conversely [**4**].

THEOREM 2.1. *Suppose Ω is P-convex but not strongly P-convex. Then*
(a) $\mathfrak{D}'(\Omega)/P^{-1}(0)$ *is not complete.*

(b) $\mathfrak{D}'(\Omega)$ *is not fully complete.*

(c) *The range of* ${}^tP: C_0^\infty(\Omega) \to C_0^\infty(\Omega)$, *although closed, is not bornological.*

PROOF. Let R denote the range of P, $N = P^{-1}(0)$ the kernel of P. By Lemma 2.1, the induced mapping $P_0: \mathfrak{D}'(\Omega)/N \to R$ is open. Hence it is an algebraic and topological isomorphism. R is dense in $\mathfrak{D}'(\Omega)$ but not equal to $\mathfrak{D}'(\Omega)$, hence not complete. This proves (a). Since any quotient of a fully complete space by a closed subspace is complete, (b) is an immediate consequence of (a). To prove (c), we note that by Lemma 2.1, ${}^tP$ has bounded inverse. Thus, if the range of ${}^tP$ were bornological, $({}^tP)^{-1}$ would be continuous, which one can easily show implies that P is surjective.

REFERENCES

1. N. Bourbaki, Livre V: *Espaces vectoriels topologiques*, Actualités Sci. Indust., no. 1230, Hermann, Paris, 1955. MR **17**, 1109.

2. C. Harvey, *On domination estimates and global existence*, J. Math. Mech. **16** (1967), 675–702. MR **35** #5750.

3. L. Hörmander, *On the range of convolution operators*, Ann. of Math. (2) **76** (1962), 148–170. MR **25** #5379.

4. ———, *Linear partial differential operators*, Die Grundlehren der math. Wissenschaften, Band 116, Academic Press, New York and Springer-Verlag, Berlin and New York, 1963. MR **28** #4221.

EMORY UNIVERSITY, ATLANTA, GEORGIA 30322

RICE UNIVERSITY, HOUSTON, TEXAS 77001