NORM CONVERGENT EXPANSIONS FOR GAUSSIAN PROCESSES IN BANACH SPACES[1]

NARESH C. JAIN AND G. KALLIANPUR

ABSTRACT. Several authors have recently shown that Brownian motion with continuous paths on [0, 1] can be expanded into a uniformly convergent (a.s.) orthogonal series in terms of a given complete orthonormal system (CONS) in its reproducing kernel Hilbert space (RKHS). In an earlier paper we generalized this result to Gaussian processes with continuous paths. Here we obtain such expansions for a Gaussian random variable taking values in an arbitrary separable Banach space. A related problem is also considered in which starting from a separable Hilbert space H with a measurable norm $\|\cdot\|_1$ defined on it, it is shown that the corresponding abstract Wiener process has a $\|\cdot\|_1$-convergent orthogonal expansion in terms of a CONS chosen from H.

1. **Introduction.** In [5] we used the results of Itô and Nisio [4] to show the uniform convergence in the time parameter of expansions of a Gaussian process. A special case of Theorem 1 of [5] is the following result:

THEOREM 1. *Let* $(X_t, t\in[0, 1])$ *be a Gaussian process with covariance* R *and having continuous paths with probability* 1 *on a probability space* $(\Omega, \mathfrak{F}, P)$. *Let* $\{e_j\}_1^\infty$ *be a complete orthonormal system (CONS) in the reproducing kernel Hilbert space (RKHS)* $H(R)$. *Then there exists a sequence* $\{\xi_j\}_1^\infty$ *of independent* $N(0, 1)$ *(Gaussian with mean* 0, *variance* 1) *random variables on* $(\Omega, \mathfrak{F}, P)$ *such that the partial sums*

$$\sum_{j=1}^{n} \xi_j(\omega)e_j(t) \tag{1.1}$$

converge uniformly in t *to* $X_t(\omega)$ *a.s. (as* $n\to\infty$).

It was pointed out in [5] that Theorem 1 implies, in particular, the a.s. uniform convergence of the Karhunen-Loève expansion of the process X_t.

Our aim here is to study similar questions in a more general context. Adopting the point of view of L. Gross [2] we obtain expansions

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similar to (1.1) for all Gaussian processes on separable Banach spaces; Theorem 1 itself being a special case of this general result.

The set up that permits us to define "orthogonal expansions" for Gaussian processes in Banach spaces is explained below in terms of the abstract Wiener processes of Gross. The precise definitions are given in the next section.

Let H be a separable Hilbert space and μ the cylinder set measure corresponding to the canonical Gaussian weak distribution F. If H is infinite dimensional (which is what we assume from now on) it is well known that μ is not countably additive. Let there be a measurable norm (in the sense of Gross) on H and let B be the separable, infinite dimensional Banach space which is the completion of H under this norm (the measurable norm is necessarily weaker than the Hilbert norm on H [2]). Then it is known [3] that μ can be extended to B to be a countably additive Gaussian measure on the topological Borel field $\mathfrak{A}(B)$ of B.

X is said to be a Gaussian, B-valued random variable (or process) on $(\Omega, \mathfrak{F}, P)$ if for every f in B^* (the dual of B), $\langle f, X\rangle$ is a real-valued Gaussian random variable with mean 0, where $\langle f, x\rangle$ denotes the scalar product of an element f of B^* and an element x of B. On account of the separability of B and Proposition 2.1 of [4], it follows that X is a map of Ω into B which is $(\mathfrak{F}, \mathfrak{A}(B))$-measurable, so that our definition of a B-valued random variable coincides with that given by Itô and Nisio in [4]. The probability measure $\mu = PX^{-1}$ defined on the σ-field $\mathfrak{A}(B)$ of Borel sets of B will be referred to as the distribution of X. The main results are the following:

THEOREM 2. *Let H be an infinite dimensional separable Hilbert space. Let F denote the canonical Gauss distribution on H. Let B be the infinite dimensional separable Banach space which is the completion of H under a given measurable norm $\|\cdot\|_1$, and let μ denote the countably additive Gaussian measure on $(B, \mathfrak{A}(B))$ which is determined by F. If $\{e_j\}_1^\infty$ is a CONS in H, then there exists a sequence $\{\xi_j\}_1^\infty$ of independent $N(0, 1)$ random variables on a probability space $(\Omega, \mathfrak{F}, P)$ such that the B-valued random variables*

$$Y_n(\omega) = \sum_{j=1}^{n} \xi_j(\omega) e_j \tag{1.2}$$

converge in $\|\cdot\|_1$ norm a.s. to a B-valued Gaussian random variable Y with distribution μ, as $n \to \infty$.

THEOREM 3. *Let B be an arbitrary separable Banach space with norm $\|\cdot\|_B$ and let X be a Gaussian, B-valued random variable defined on a* *probability space* $(\Omega, \mathfrak{F}, P)$. *Let* μ *denote the distribution of* X, *i.e. the probability measure in* B *induced by* X. *Then there exists a separable Hilbert space* H *contained in* B *such that the following is true*:

(1.3) $\|\cdot\|_B$ *is a measurable norm on* H.

If $\overline{H}$ *denotes the closure of* H *in* B, *then*

(1.4) $\overline{H}$ = *topological support of* μ *in* B.

(1.5) *To every choice of CONS* $\{e_j\}_1^\infty$ *in* H *there corresponds a sequence* $\{\xi_j\}_1^\infty$ *of independent* $N(0, 1)$ *random variables on* $(\Omega, \mathfrak{F}, P)$ *with the property that*

$$X(\omega) = \sum_{j=1}^{\infty} \xi_j(\omega) e_j \quad a.s.\ (P), \tag{1.6}$$

the convergence of the series on the right-hand side being in the sense of the norm $\|\cdot\|_B$. *The series in* (1.6) *might be regarded as the orthogonal expansion of* X *corresponding to* $\{e_j\}_1^\infty$.

2. **Preliminaries.** The notions of a cylinder set measure and a weak distribution on a locally convex, linear topological space L are equivalent. Let L^* denote the topological dual of L.

Definition 2.1. A weak distribution on L is an equivalence class of linear maps F from L^* to the linear space $M(\Omega, \mathfrak{F}, P)$ of random variables on some probability space $(\Omega, \mathfrak{F}, P)$ (the choice of which depends on F).

Definition 2.2. If $L = H$, a separable Hilbert space, a weak distribution F is called a canonical Gauss distribution on H if to each $h \in H^*$ the real random variable $F(h)$ is normally distributed with mean 0 and variance $\|h\|_H^2$. ($\|\cdot\|_H$ denotes the Hilbert norm on H^*.)

From now on F will be a representative of the canonical Gauss distribution.

A function f is said to be a tame function on L if $f(x) = \phi[\langle y_1, x\rangle, \cdots, \langle y_n, x\rangle]$, where $y_1, \cdots, y_n \in L^*$ and ϕ is a Baire function of n variables; $\langle y, x\rangle$ denotes the value of y at x. For $\omega \in \Omega$ the random variable $f^\sim(\omega) = \phi[F(y_1)(\omega), \cdots, F(y_n)(\omega)]$ has the same probability distribution as f under the weak distribution F.

Remark. Let B be a Banach space with norm $\|\cdot\|_B$ and let $H \subset B$ be a Hilbert space with norm $\|\cdot\|_H$ and inner product $(\cdot, \cdot)_H$. Let $e_1, \cdots, e_n \in H$. Then the function $u(a_1, \cdots, a_n) = \|a_1e_1 + \cdots + a_ne_n\|_B$ is a continuous function in n variables, and hence a Baire function. It thus follows that for $x \in H$ the function $g(x) = \|(e_1, x)_He_1 + \cdots + (e_n, x)_He_n\|_B$ is a tame function on H and

$$g^\sim(\omega) = \|F(e_1)(\omega)e_1 + \cdots + F(e_n)(\omega)e_n\|_B. \tag{2.1}$$

This fact will be needed later.

Let H be a separable Hilbert space with the canonical Gauss distribution F and $\mathcal{P}$ the family of all finite dimensional projections Q on H.

DEFINITION 2.3 [2]. A norm $\|x\|_1$ on H is said to be measurable (with respect to F) if to every $\epsilon>0$ there exists a projection $Q_\epsilon \in \mathcal{P}$ such that

(2.2) $\operatorname{Prob}\{\|Qx\|_1^\sim > \epsilon\} < \epsilon$ for all $Q \perp Q_\epsilon$, $Q \in \mathcal{P}$.

DEFINITION 2.4. Let R be a continuous covariance function on $[0, 1]\times[0, 1]$. Then the reproducing kernel Hilbert space (RKHS) of R, denoted by $H(R)$, is a Hilbert space of continuous functions f on $[0, 1]$ with the following properties:

(2.3) $R(\cdot, t) \in H(R)$, for all $t \in [0, 1]$,

(2.4) $(f, R(\cdot, t))_{H(R)} = f(t)$ for $t \in [0, 1]$,

where $(\cdot, \cdot)_{H(R)}$ denotes the inner product in $H(R)$.

We will need the following result of Gross [2] in proving Theorem 2.

LEMMA 2.1 [2, COROLLARY 5.2]. *Let Q_α and Q_β be two nets of finite dimensional projections on a Hilbert space H, each converging to the identity, and let $\|\cdot\|_1$ be a measurable norm on H. Then the random variables $\|Q_\alpha x\|_1^\sim$ converge in probability to a random variable $\|x\|_1^\sim$. Furthermore $\|(Q_\alpha - Q_\beta)x\|_1^\sim$ converges to zero in probability as $\alpha, \beta \to \infty$.*

3. **Proof of Theorem 2.** Let $\{e_j\}_1^\infty$ be a CONS in H. If F is the canonical Gauss distribution, the random variables $F(e_j)$, $j\geq 1$, are independent $N(0, 1)$ defined over some probability space $(\Omega, \mathfrak{F}, P)$. Define $\xi_j(\omega) = F(e_j)(\omega)$ for $\omega \in \Omega$. Let $Y_n(\omega)$ be the partial sums defined in (1.2). Let μ_n denote the distribution of Y_n, i.e. the Gaussian measure on $(B, \mathfrak{A}(B))$ induced by Y_n. Let $\{Y_{n(k)}\}$ be an arbitrary but fixed subsequence of $\{Y_n\}$. Let Q_n denote the projection on H given by $Q_n x = \sum_{j=1}^n (x, e_j)e_j$ for $x \in H$, where $(\cdot, \cdot)$ is the inner product in H. By Lemma 2.1 we can find $\{n(k)'\} \subset \{n(k)\}$ such that for $k \geq 1$

$$P\{\|(Q_{n(k+1)'} - Q_{n(k)'})x\|_1^\sim \geq 1/2^k\} \leq 1/2^k. \tag{3.1}$$

This can be seen as follows: by Lemma 2.1 one can pick $\{n(k)'\} \subset \{n(k)\}$, such that $n(k)' < n(k+1)'$ and for $m,n \geq n(k)'$, $P\{\|(Q_m - Q_n)x\|_1^\sim \geq 1/2^k\} \leq 1/2^k$. For this sequence $\{n(k)'\}$ it is now clear that (3.1) holds. Now $\|(Q_{n(k+1)'} - Q_{n(k)'})x\|_1 = \|\sum_{j=n(k)'+1}^{n(k+1)'} (x, e_j)e_j\|_1$ and as we already explained in the previous section we have

$$\left\|\sum_{j=n(k)'+1}^{n(k+1)'} (x, e_j)e_j\right\|_1^\sim(\omega) = \left\|\sum_{j=n(k)'+1}^{n(k+1)'} \xi_j(\omega)e_j\right\|_1.$$

Hence (3.1) can be written as

$$P\{\|Y_{n(k+1)'} - Y_{n(k)'}\|_1 \geqq 1/2^k\} \leqq 1/2^k.$$

It is thus clear that $\{Y_{n(k)'}\}$ converges in probability to a random variable and consequently by Theorem 3.1 [4] $\{\mu_{n(k)'}\}$ converges in the Prohorov metric to a probability measure, say μ'. It is easy to see that $\mu' = \mu$, for if $f \in B^*$ and u is real, then

$$\begin{aligned}\int_B \exp(iu\langle f, x\rangle)\mu'(dx) &= \lim_{n\to\infty} \int_B \exp(iu\langle f, x\rangle)\mu_n(dx) \\ &= \lim_{n\to\infty} \int_\Omega \exp(iu\langle f, Y_n\rangle)P(d\omega) \\ &= \lim_{n\to\infty} \exp\left(-\frac{u^2}{2}\sum_{j=1}^n \langle f, e_j\rangle^2\right) \\ &= \exp\left(-\frac{u^2}{2}(f, f)\right)_H.\end{aligned}$$

Hence if $f \in B^*$ $(\subset H^*)$, then $\langle f, x\rangle$ is $N(0, \|f\|_H^2)$. Thus μ and μ' have the same cylinder set measure and therefore must be identical. We have thus shown that every subsequence of $\{\mu_n\}$ has a further subsequence which converges in the Prohorov metric to the probability measure μ. Hence $\{\mu_n\}$ itself converges to μ in this metric. Theorem 3.1 [4] now shows that Y_n converges a.s. in $\|\cdot\|_1$ norm to a random variable Y. It is clear that μ is the distribution of Y.

4. **Proof of Theorem 3.** Let us first assume that $B = C_0$, a closed linear subspace of $C[0, 1]$. Let $\mathfrak{A}(C)$ denote the Borel sets of $C[0, 1]$ and $\mathfrak{A}(C_0)$ those of C_0. X induces a Gaussian measure μ on $(C_0, \mathfrak{A}(C_0))$. For $\omega \in \Omega$ define $X_t(\omega) = [X(\omega)](t)$, $0 \leqq t \leqq 1$. Then $\{X_t, 0 \leqq t \leqq 1\}$ is a Gaussian process on the probability space $(\Omega, \mathfrak{F}, P)$ with continuous paths. Let $H(R)$ be its RKHS. Since the random variables X_t are Gaussian and the process has continuous paths it follows easily that R is continuous and hence $H(R)$ consists of continuous functions. We now observe that from Theorem 6 of [6] the sup-norm closure of $H(R)$, denoted as $[H(R)]_1$, must be contained in C_0. Theorem 3 is now true with $B = C_0$ and $H = H(R)$. To see this note that conclusion (1.3) follows from Theorem 7 of [6] according to which the sup-norm must be a measurable norm on $H(R)$ if the process has continuous paths. Theorem 6 of [6] yields (1.4) and the remaining assertions follow from Theorem 1. It now remains to consider the case when B is a separable Banach space. Let $\theta: B \to C_0$, where C_0 is a closed subspace of $C[0, 1]$, be an isometric isomorphism from B onto C_0 (one is

constructed in [1, p. 186]). Let $Y=\theta \circ X$ be the C_0-valued random variable on $(\Omega, \mathfrak{F}, P)$. Then by the special case we have considered there is a Hilbert space $H(R)\subset C_0$ such that $[H(R)]_1$, the closure of $H(R)$ in C_0, is the topological support of μ, where μ is the Gaussian measure on C_0 induced by Y. If we let $H=\theta^{-1}(H(R))$ and define the inner product (x, y) in H to be $(\theta x, \theta y)_{H(R)}$, where $(\cdot, \cdot)_{H(R)}$ denotes the inner product in $H(R)$, then H becomes a Hilbert space with inner product (x, y). θ is thus not only the isometric isomorphism between B and C_0, but it also preserves the inner product between the Hilbert spaces H and $H(R)$. It is now straightforward to check that $\|\cdot\|_B$ is a measurable norm in H (simply because the sup-norm is a measurable norm in $H(R)$). This H now serves the purpose of the Hilbert space in the statement of Theorem 3. (1.4) follows from the special case by observing that θ is an isometric isomorphism. Let $\{e_j\}_1^\infty$ be any CONS in H. Then $\{\theta e_j\}_1^\infty$ is a CONS in $H(R)$ and there exist independent $N(0, 1)$ random variables $\{\xi_j\}_1^\infty$ on $(\Omega, \mathfrak{F}, P)$ such that $\sum_{j=1}^n \xi_j(\omega)\theta(e_j)$ converges in the sup-norm a.s. to $\theta \circ X(\omega)$. Again, because θ is an isometric isomorphism between B and C_0 it now follows that $\sum_{j=1}^n \xi_j(\omega)e_j$ converges in $\|\cdot\|_B$ norm a.s. to $X(\omega)$. This establishes (1.5) and (1.6) in the general case and the proof is complete.

From the proof just given, it is clear that Theorem 1 [5] is contained in the more abstract version given in Theorem 3.

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University of Minnesota, Minneapolis, Minnesota 55455