CONCERNING PRODUCT INTEGRALS AND EXPONENTIALS

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ABSTRACT. Suppose S is a linearly ordered set, N is the set of real numbers, G is a function from $S\times S$ to N, and all integrals are of the subdivision-refinement type. We show that if $\int_a^b G^2=0$ and either integral exists, then the other exists and ${}_a\prod^b(1+G)=\exp\int_a^b G$. We also show that the bounded variation of G is neither necessary nor sufficient for $\int_a^b G^2$ to be zero.

B. W. Helton, J. S. MacNerney, and H. S. Wall have established various relationships between integral equations, sum integrals, and product integrals. This paper establishes a relationship between exponentials, sum integrals, and product integrals which may be used to evaluate certain product integrals or sum integrals. Integrals used are of the subdivision-refinement type and complete definitions of these and other terms and symbols used in this paper may be found in [1] or [2]. Suppose S is a linearly ordered set [2] and N is the set of real numbers. All functions considered will be functions from $S\times S$ to N unless otherwise noted. In [1, Theorem 3.4] it is shown that for functions of bounded variation from $S\times S$ to N the following two statements are equivalent: (1) $\int_a^b G$ exists and (2) ${}_a\prod^b(1+G)$ exists. Under the hypothesis that $\int_a^b G^2=0$, we show that the following two statements are equivalent for functions from $S\times S$ to N: (1) $\int_a^b G$ exists and (2) ${}_a\prod^b(1+G)$ exists and is not zero. It is also noted that neither of the following two statements is a consequence of the other. (1) $\int_a^b G^2=0$ and (2) G is of bounded variation on $[a, b]$.

THEOREM 0. *If* ${}_a\prod^b(1+G)$ *exists and is not zero then if* $\epsilon>0$ *there is a subdivision* D *of* $\{a, b\}$ *such that if* $D'=\{x_i\}_{i=0}^n$ *is a refinement of* D, *then*

$$\left|\log\frac{{}_a\prod^b(1+G)}{\prod_{D'}(1+G_i)}\right|<\epsilon.$$

The proof of this theorem is omitted.

THEOREM 1. *Neither of the following statements is a consequence of the other*:

Received by the editors December 11, 1969.

AMS Subject Classifications. Primary 2645, 2649; Secondary 4513.

Key Words and Phrases. Exponentials, product integrals, subdivision-refinement type integrals, bounded variation.

(1) $\int_a^b G^2 = 0$.

(2) *G is of bounded variation.*

INDICATION OF PROOF. Let G be the function such that for each $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$,

$$\begin{aligned} G(x,y) &= x, \quad x = 1/n, \quad n \text{ an integer, and } |x-y| \geqq 1/n - 1/(n+1), \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

$\int_0^1 G^2 = 0$ but G is not of bounded variation on $[0, 1]$ and $\int_0^1 G$ does not exist. Hence (2) is not a consequence of (1).

Let H be the function such that for each $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$,

$$\begin{aligned} H(x,y) &= 1, \quad x = 0, \quad y > x, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

$V_0^1 H = 1$ but $\int_0^1 H^2 = 1$. Hence (1) does not follow from (2).

The following theorem may be found in [2, p. 151] and may be established by induction.

THEOREM 2. *If n is an integer greater than 1 and each of $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ is a sequence of numbers, then*

$$\prod_{i=1}^n A_i - \prod_{i=1}^n B_i = \sum_{i=1}^n \left(\prod_{j=1}^{i-1} B_j \right) (A_i - B_i) \left(\prod_{k=i+1}^n A_k \right).$$

THEOREM 3. *If $\int_a^b G^2 = 0$, then the following two statements are equivalent*: (1) *$\int_a^b G$ exists.*

(2) *${}_a\prod^b (1+G)$ exists and is not zero. Furthermore, if either* (1) *or* (2) *is true, then $\int_a^b G = \log {}_a\prod^b (1+G)$.*

PROOF. 1. Suppose (1) is true and $\epsilon > 0$. Since $\int_a^b G^2 = 0$ and $\int_a^b G$ exist then there is a subdivision D of $\{a, b\}$ such that if D' is a refinement of D, then there is a number k such that:

$$\sum_{D'} G_i^2 < \frac{1}{4} \quad \text{and hence } |G_i| < \frac{1}{2}, \tag{1}$$

$$\sum_{D'} G_i^2 < \frac{\epsilon}{2 \exp\left(\frac{3}{2} + \int_a^b G\right)}, \tag{2}$$

$$|k| < \frac{\epsilon}{8 \exp\left(\frac{3}{2} + \int_a^b G\right)}, \tag{3}$$

(4) $|k| < \frac{1}{2}$, so if $n > m \geqq 0$,

$$\exp(mk/n) < \exp(\tfrac{1}{2}) \quad \text{and} \quad \exp(-k) < \exp(\tfrac{1}{2}),$$

and

$$\int_a^b G = \sum_{D'} G_i + k. \tag{5}$$

Let $D' = \{x_i\}_{1=0}^n$ be a refinement of D.

$$\begin{aligned}
\sum_{i=1}^n & \left| \exp\left(G_i + \frac{k}{n}\right) - G_i - 1 \right| \\
&= \sum_{i=1}^n \left| -1 - G_i + \sum_{j=0}^\infty \frac{(G_i + k/n)^j}{j!} \right| \\
&\leqq \sum_{i=1}^n \left| \frac{k}{n} \right| + \sum_{i=1}^n \left| \sum_{j=2}^\infty \frac{(G_i + k/n)^j}{j!} \right| \\
&\leqq |k| + \sum_{i=1}^n (G_i + k/n)^2 \cdot \left(\sum_{j=2}^\infty \frac{1}{j!} \right) \\
&< |k| + \sum_{i=1}^n (G_i + k/n)^2 \\
&\leqq |k| + \frac{\epsilon}{2 \exp\left(\frac{3}{2} + \int_a^b G\right)} + |k| + |k| \\
&< \frac{7\epsilon}{8 \exp\left(\frac{3}{2} + \int_a^b G\right)} .
\end{aligned}$$

Therefore,

$$\sum_{i=1}^n |\exp(G_i + k/n) - G_i - 1| < \frac{7\epsilon}{8 \exp\left(\frac{3}{2} + \int_a^b G\right)} .$$

Then,

$$\left|\prod_{i=1}^{n}(1+G_i)-\exp\left(\int_a^b G\right)\right|$$

$$=\left|\prod_{i=1}^{n}(1+G_i)-\prod_{i=1}^{n}\exp(G_i+k/n)\right|$$

$$\leqq \sum_{i=1}^{n}\left|\prod_{j=1}^{i-1}(1+G_i)\right|\cdot\left|\exp(G_i+k/n)-1-G_i\right|\cdot\left|\prod_{j=i+1}^{n}\exp(G_i+k/n)\right|$$

$$\leqq \sum_{i=1}^{n}\left|\prod_{j=1}^{i-1}\exp G_i\right|\cdot\left|\prod_{j=i+1}^{n}\exp(G_i+k/n)\right|\cdot\left|\exp(G_i+k/n)-1-G_i\right|$$

$$=\sum_{i=1}^{n}\left|\exp\left(\sum_{j=1}^{n}G_j+k-G_i-k+((n-i)/n)k\right)\right| \cdot\left|\exp(G_i+k/n)-1-G_i\right|$$

$$<\sum_{i=1}^{n}\exp\left(\int_a^b G\right)\cdot\exp(\tfrac{1}{2})\cdot\exp(\tfrac{1}{2})\cdot\exp(\tfrac{1}{2})\cdot\left|\exp(G_i+k/n)-1-G_i\right|$$

$$<\exp\left(\int_a^b G+\frac{3}{2}\right)\frac{7\epsilon}{8\exp\left(\int_a^b G+\frac{3}{2}\right)}$$

$$<\epsilon.$$

Hence, $\left|\prod_{i=1}^{n}(1+G_i)-\exp(\int_a^b G)\right|<\epsilon$ so that ${}_a\prod^b(1+G)$ exists and is $\exp(\int_a^b G)$.

2. Suppose (2) is true and $\epsilon>0$. Since $\int_a^b G^2=0$, ${}_a\prod^b(1+G)$ exists and is not zero, then there exists a subdivision D of $\{a, b\}$ such that if D' is a refinement of D, then

(1) $$|G_i| < \tfrac{1}{2}$$

(2) $$\left|\log\frac{{}_a\prod^b(1+G)}{\prod_{D'}(1+G_i)}\right|<\frac{\epsilon}{2}$$

(3) $$\log(1+G_i)=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}G_i^j}{j}$$

(4) $$M=\sum_{j=2}^{\infty}\frac{(\frac{1}{2})^{j-2}}{j}\geqq\sum_{j=2}^{\infty}\frac{|(G_i)^{j-2}|}{j}$$

(5) $$\sum_{D'}G_i^2<\frac{\epsilon}{2M}\cdot$$

Let $D'=\{x_i\}_{i=0}^{n}$ be a refinement of D, then

$$\left|\log {}_a\prod^b (1+G) - \sum_{i=1}^n G_i\right|$$

$$\leqq \left|\log \prod_{i=1}^n (1+G_i) - \sum_{i=1}^n G_i\right| + \left|\log \frac{{}_a\prod^b (1+G)}{{}_{D'}\prod (1+G_i)}\right|$$

$$< \left|\sum_{i=1}^n [\log(1+G_i) - G_i]\right| + \frac{\epsilon}{2}$$

$$= \left|\sum_{i=1}^n \left[\sum_{j=1}^\infty (-1)^{j-1} \frac{G_i^j}{j} - G_i\right]\right| + \frac{\epsilon}{2}$$

$$= \left|\sum_{i=1}^n \sum_{j=2}^\infty (-1)^{j-1} \frac{G_i^j}{j}\right| + \frac{\epsilon}{2}$$

$$= \left|\sum_{i=1}^n \left[G_i^2 \cdot \sum_{j=2}^\infty (-1)^{j-1} \frac{G_i^{j-2}}{j}\right]\right| + \frac{\epsilon}{2}$$

$$\leqq \sum_{i=1}^n \left[G_i^2 \cdot \sum_{j=2}^\infty \frac{|G_i|^{j-2}}{j}\right] + \frac{\epsilon}{2}$$

$$\leqq M \sum_{i=1}^n G_i^2 + \frac{\epsilon}{2} < M \cdot \frac{\epsilon}{2M} + \frac{\epsilon}{2} = \epsilon.$$

Hence,

$$\left|\log {}_a\prod^b (1+G) - \sum_{i=1}^n G_i\right| < \epsilon$$

so that $\int_a^b G$ exists and is $\log {}_a\prod^b (1+G)$.

REMARK. As noted by the referee, a function G from $S \times S$ to N may have the property that $\int_a^b G^2 = 0$ and $\int_a^b G$ exists yet G fails to be of bounded variation on $[a, b]$. As an example of such a function we offer the following: Suppose for $0 < x \leqq 1$, $g(x) = x \sin(\pi/x)$ and $g(0) = 0$ and for each $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, $G(x, y) = g(y) - g(x)$. $\int_0^1 G^2 = \int_0^1 G = 0$, but $\int_0^1 |G|$ does not exist.

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